On the Teaching of the Theory of Vectors Spaces in First Year of French Science University

Jean-Luc Dorier

Abstract: In the French tradition of Bourbaki, the theory of vector spaces is usually presented in a very formal setting, which causes severe difficulties to many students. The aim of this paper is to analyze the underlying reasons of these difficulties and to suggest some ways to make the first teaching of the theory of vector spaces less inefficient for many students. We do not reject the necessity for formalism. On the contrary, on the basis of a historical analysis we can explain the specific meaning it has in the theory. From this mathematical analysis with a historical perspective, we analyze the teaching and the apprehension of vector space theory in a new approach. For instance, we will show that mistakes made by many students can be interpreted as a result of a lack of connection between the new formal concepts and their conceptions previously acquired in more restricted, but more intuitively based areas. Our conclusions will not plead for avoiding formalism but for a better positioning of the formal concepts with regard to previous knowledge of the students as well as special care to be given in making the role and the meaning of formalism in linear algebra explicit to the students.

Introduction

In collaboration with A. Robert, J.Robinet, and M. Rogalski, we have developed a research program on the learning and teaching of linear algebra in the first year of French science universities. This work[1], which started some ten years ago, includes the elaboration and evaluation of experimental teaching based on a substantial historical study and a theoretical approach within the French context of "didactique des mathmeatiques"(the French name for a field of research which would more or less correspond to mathematical education research in the USA). In this paper, we will try to summarize the main issues of our research, focusing the concepts of linear dependence and independence.

If a system of linear equations has as many equations as unknowns (n), dependence between the equations of the system may be understood in different ways. If one is not familiar with the notion of linear dependence, but more concerned with solving the system, the dependence reflects an indetermination on the solutions of the system. Practically, it means that, in the process of resolution, one (or more) unknown(s) will be left undetermined. Therefore n

[1] The results of this work are gathered in a book (Dorier1997) which also includes the presentation of other works (in France, Canada, USA and Morocco).

dependent equations in *n* unknowns will be characterized by the fact that they determine unknowns less than *n* and thus act as if they were less than *n*. With regard to the solving of equations, dependence is therefore revealed by an accident in the solving these results in the vanishing of at least on equation and the indetermination of at least one of the unknowns. It is an accident because *n* equations usually determine *n* unknowns exactly. If the method for solving the system uses linear combinations, this accident may be connected to the fact that a linear combination of the equations is zero. If the dependence is "obvious", one may even see directly that one equation is a linear combination of the others, although this will not be the central characteristic of the dependence.

Historical background

Although it might be difficult to admit for a modern mathematician so familiar with the vocabulary and basic notions of linear algebra, the above way of considering dependence between equations may be found (with the same words) in a text by Euler dating from 1750. It still prevailed in most of the texts about linear equations up to the end of the 19th century (see Dorier 1993, 1995a, 1996b). Euler's text is the first in which the question of dependence was discussed. The general idea that n equations determine n unknowns was so strong that nobody had taken the pain to discuss the exceptional case, until Euler was confronted with Cramer's paradox and pointed out this particularity.

He starts by an example with two equations:

Let us just look at these two equations 3x -2y = 5 *and* 4y = 6x -10, *one will see immediately that it is not possible to determine the two unknowns* x *and* y, *as if one eliminates* x , *then the other unknown* y *disappears by itself and one gets an identical equation, from which it is not possible to determine anything. The reason for this accident is quite obvious as the second equation can be changed into* 6x -4y = 10, *which being simply the first one doubled, is thus not different.*[2]

It is clear -especially by reading the end of this quotation- that Euler does not intend to fool his reader, even though he artificially hides the similarity of the two equations. Yet, it is also clear that it is not the fact that the two equations are similar that determines the dependence of the equations, but the fact that something unusual -an accident- happens in the final step of the solving process. This accident reveals the dependence of the equations, because, although there

[2] *On n'a qu'à regarder ces deux équations* 3x- 2y = 5 *et* 4y = 6x -10 *et on verra d'abord qu'il n'est pas possible d'en déterminer les deuxinconnues* x *et* y, *puisqu'en éliminant l'une* x, *l'autres'en va d'elle-même et on obtient une équation identique, dont on esten état de ne déterminer rien. La raison de cet accident saute d'abord aux yeux puisque laseconde équation se change en* 6x - 4y = 10, *qui n'étant que lapremière doublée, n'en differe point.* [Euler 1750, 226]

are two of them, these equations do not determine two unknowns. Mathematically speaking, the two statements are logically connected, a linear dependence between *n* equations in *n* unknowns is equivalent to the fact that the system will not have a unique solution; However the two properties correspond to two different conceptions of dependence. To be able to distinguish these two conceptions, I will call Euler's conception, *inclusive dependence*. I wish to insist on the fact that this conception is natural in the context in which Euler and all the mathematicians of his time were working, that is to say with regard to the solving of linear equations, and not the study of equations as objects on their own.

Let us now see what Euler says for three equations:

[...] *The first one, being not different from the third one, does not contribute at all in the determination of the three unknowns.*

But there is also the case when one of the three equations is contained in the two others. [...] So when it is said that to determine three unknowns, it is sufficient to have three equations, it is necessary to add the restriction that these three equations are so different that none of them is already comprised in the others.[3]

It is important to notice that, for three equations, Euler separates the case when two equations are equal from the case when the three equations are globally dependent. This points out the intrinsic difficulty of the concept of dependence which has to take all the equations in a whole system into account, and not only the relations in pairs. We will see that students have real difficulties with this point. On the other hand, Euler's use of terms such as *comprised* or *contained*, refers to the conception of inclusive dependence as we explained above. It does not mean that Euler was not aware of the logical equivalence with linear dependence, but, within his practice with linear equations, the conception of inclusive dependence is more consistent and efficient. Yet, there is a difficulty for further development; indeed, the conception of inclusive dependence is limited to the context of equations and cannot be applied to other objects like *n*-tuples for instance. Therefore inclusive dependence is context-dependent although linear dependence is a general concept that applies to any object of a linear structure. Yet, in his text Euler was able to bring out issues that can be considered in many aspects as the first consistent ideas on the concept of

---

[3] [...]*La première ne différant pas de la troisième,ne contribue en rien à la détermination des trois inconnues.*

Mais il y a aussi le cas, où unedes trois équations est contenue dans les deux autres conjointement[...] Ainsi quand on dit que pour déterminer trois inconnues, il suffitd'avoir trois équations, il y faut rajouter cette restriction, que ces trois équations diffèrent tellement entr'elles, qu'aucune ne soit déjà contenuedasn les deux autres. [ibid., 226] rank. Indeed, he discusses, although in a very intuitive and vague manner, the relation between the size of the set of solutions and the number of relations of dependence between equations. We will see now that it took more than a century for the concept of rank to come to maturity.

1750 is also the year Cramer published the treatise that introduced the use of determinants which was to dominate the study of linear equations until the first quarter of the 20th century. In this context, dependence was characterized by the vanishing of the determinant. The notion of linear dependence, now basic in modern linear algebra, did not appear in its modern form until 1875. Frobenius introduced it, pointing out the similarity with the same notion for n-tuples. He was therefore able to consider linear equations and n-tuples as identical objects with regard to linearity. This simple fact may not seem very relevant but it happened to be one of the main steps toward a complete understanding of the concept of rank. Indeed in the same text, Frobenius was able not only to define what we would call a basis of solutions but he also associated a system of equations to such a basis (each n-tuple is transformed into an equation). Then he showed that any basis of solutions of this associated system has an associated system with the same set of solutions as the initial system. This first result on duality infinite-dimensional vector spaces showed the double level of invariance connected to rank both for the system and for the set of solutions. Moreover, Frobenius' approach allowed a system to be seen as an element of a class of equivalent systems having the same set of solutions: a fundamental step toward the representation of sub-spaces by equations.

This brief summary of over a century of history[4] shows how adopting a formal definition (here of linear dependence and independence) may be a fundamental step in the construction of a theory, and is therefore an essential intrinsic constituent of this theory.

Didactical issues

Anyone who has taught a basic course in linear algebra knows how difficult it may be for a student to understand the formal definition of linear independence, and to apply it to various contexts. Moreover, once students have proven their ability to check whether a set of n-tuples, equations, polynomials or functions are independent, they still may not be able to use the concept of linear independence in more formal contexts.

A. Robert and J. Robinet (1989) have tested beginners on the following

[4] For more details see (Dorier 1993, 1995a and1997).

questions: [5]

1. Let U, V and W be three vectors in IR^3. If any pair of them is non-collinear, are they independent?

2.1. Let U, V and W be three vectors in IR^3, and f a linear operator in IR^3. If U, V and Ware independent, are f(U), f(V) and f(W) independent ?

2.2. Let U, V and W be three vectors in IR^3, and f a linear operator in IR^3. If f(U), f(V)and f(W) are independent, are U, V and W independent ?

Beginners generally failed these questions. In the three cases, they used the formal definition of linear independence and tried different combinations with the hypotheses and the conclusions leading to apparently erratic proofs, that teachers usually reject without further comment.

For instance, to the first question a majority of students answered "yes" giving proofs that are examples of the difficulty they have in treating linear (in)dependence globally. Indeed, many students are inclined to treat the question of linear (in)dependence by successive steps, starting with two vectors, and then introducing the others one by one much like Euler did. We will say that they have a local approach to a global question. Indeed, in many cases, at least if it is well controlled, this approach may be correct and actually quite efficient, yet, it is a source of mistakes in several situations. The students have built themselves what G. Vergnaud (1990) calls *théorèmes-en-acte* (i.e. rules of action or theorems that are valid in some restricted situations but create mistakes when abusively generalized to more general cases). Here is a non-exhaustive list of *théorèmes-en-acte* connected with the local approach of linear (in)dependence, that we have noticed in students' activities:

- if U and V are independent of W, then U, V and W are globally independent
- if U_1 is not a linear combination of $U_2, U_3, \ldots, U_k$, then $U_1, U_2, \ldots, U_k$ are independent
- if U_1, V_1 and V_2 are independent and if U_2, V_1 and V_2 are independent, U_1, U_2, V_1 and V_2 are independent.[6]

The historical analysis confirms the fact that there is a difficulty in treating the concept of linear (in)dependence as a global property(remember the distinction made by Euler for three equations). It follows that special care must be taken in the teaching regarding this point. For instance the exercise above can be discussed with the students. Moreover, the teacher, knowing the type of *théorèmes-en-acte*, that students may have built, must help them in

[5] They are not original exercises, but they reveal important recurrent mistakes of the students.

[6] For instance when asked what is the intersection of the two subspaces generated by U_1, U_2, and V_1, V_2, students prove that neither U_1 nor U_2, are a linear combination of V_1 and V_2, and conclude that the intersection is reduced to 0.

understanding their mistakes and thereby correct them more efficiently.

To questions 2.1 and 2.2 above, many students answered respectively "yes" and "no", despite coming close to writing the correct proof for the correct answers. Here is a reconstructed proof that reflects the difficulties of the students:

If aU + bV +gW = 0 then f(aU + bV + gW) = 0
so f being a linear operator: af(U) + bf(V) +gf(W) = 0 ,
now as U, V and W are independent, then a =b=g =0,
so f(U), f(V) and f(W) are independent.

In their initial analysis, A. Robert and J. Robinet concluded that this type of answer was revealing a bad use of mathematical implication as characterized by the confusion between hypothesis and conclusion. This is indeed a serious difficulty in the use of the formal definition of linear independence. In the following year, we tested the validity of this hypothesis with different students. Before the course, we set up a test to evaluate the students' ability in elementary logic and particularly in the use of the mathematical implication (Dorier 1990a and b), and after the course, we gave the same exercise as above to the students. The results showed that the correlation was insignificant, in some cases it was even negative. Yet, on the whole (both tests included many questions), there was quite a good correlation between the two tests. This shows that if a certain level of ability in logic is necessary to understand the formalism of the theory of vector spaces, general knowledge, rather than specific competence is needed. Furthermore, if some difficulties in linear algebra are due to formalism, they are specific to linear algebra and have to be overcome essentially in this context.

On the other hand, some teachers may argue that, in general, students have many difficulties with proof and rigor. Several experiments that we have made with students showed that if they have connected the formal concepts with more intuitive concepts, then they are in fact able to build very rigorous proofs. In the case of the preceding exercise for instance, after the test, if you ask the students to illustrate the result with an example in geometry, they usually realize very quickly that there is something wrong. It does not mean that they are able to correct their wrong statement, but they know it is not correct. Therefore one main issue in the teaching of linear algebra is to give our students better ways of connecting the formal objects of the theory with their previous conceptions, in order to have a better intuitively based learning. This implies not only giving examples but also to show how all these examples are connected and what the role of the formal concepts is with regard to the mathematical activity involved.

For instance, R. Ousman (1996) gave a test to students in their final year of the *lycée* (just before entering university). Through this test, he wanted to analyze the conception of students on dependence in the context of linear equations and in geometry before the teaching of the theory of vector spaces. He gave several examples of systems of linear equations and asked the students whether the equations were independent or not. Of course he noticed mistakes due to a local approach but the answers showed also that the students justify their answer through the solving of the system. Therefore they very rarely give a justification in terms of linear combinations but most of the time in terms of equations vanishing or unknowns remaining undetermined. Their concept of (in)dependence is, like Euler's, that of inclusive dependence and not linear dependence. Yet, this is not surprising, as these students, like Euler and the mathematicians of his time, are only concerned with solving a linear system, therefore inclusive dependence is more natural and more relevant for them.

However, the formal concept is the only means to comprehend all the different types of "vectors" in the same uniform manner, as subject to linear combinations. In other words, students must be aware of the unifying and generalizing nature of the formal concept. In our research, we used what we called the *meta lever*. Therefore we built teaching situations leading students to reflect on the nature of the concepts with explicit reference to their previous knowledge (Dorier 1991,1992, 1995b and 1997 and Dorier et al. 1994a and b). In this approach, the historical analysis is a source of inspiration as well as a means of control. Nevertheless, these activities must not only involve a lecture by the teacher, nor a reconstruction of the historical development, but take into account the specific constraints of the teaching context, to reconstruct an evolution of the concepts with consistent meaning.

For instance, with regard to linear (in)dependence, French students entering university normally have a good practice of Gaussian elimination for solving systems of linear equations. It is therefore possible to begin teaching linear algebra by making them reflect on this technique not only as a tool but also as a means to investigate the properties of the systems of linear equations. This does not conform to the historical development, as the study of linear equations was historically mostly held within the theory of determinants. Yet, Gaussian elimination is a much less technical tool and a better way for showing the connection between inclusive dependence and linear dependence as identical equations (in the case when the equations are dependent) are obtained by successive linear combinations of the initial equations. Moreover, this is a context in which such question as "what is the relation between the size of the set of solutions of a homogeneous system and the number of relations of dependence between the equations?" can be investigated with the students as a first intuitive approach for the concept of rank. M. Rogalski has experimented with teaching sequences illustrating these ideas (Rogalski 1991, Dorier et al. 1994a and b and Dorier 1992 and 1997).

Finally we give the scheme of a teaching experiment that we have set up for the final step in the teaching when introducing the formal theory after having made as many connections as possible with previous knowledge and conceptions in order to build better intuitive foundations.

After the definitions of vector space, sub-space and linear combination, the notion of generator is defined. A set of generators gathers all the information we have on the sub-space, it is therefore interesting to reduce it to the minimum. The question thus is to know when it is possible to take away one generator, with the remaining vectors still being generators for the whole sub-space. The students easily find that the necessary and sufficient condition is that the vector that can be taken away must be a linear combination of the others. This provides the definition of linear dependence: "a vector is linearly dependent on others if and only if it is a linear combination of them". This definition is very intuitive, yet it is not completely formal, and it needs to be specified for sets of one vector. Without difficulty it induces the definition of a set of independent vectors as a set in which no vector is a linear combination of the others. To feel the need for a more formal definition, one just has to reach the application of this definition. Indeed, students must answer the question: "are these vectors independent or not?". With the definition above, they need to check that each vector, one after the other, is a linear combination of the others. After a few examples, with at least three vectors, it is easy to explain to the students that it would be better to have a definition in which all the vectors play the same role. One is now ready to transform the definition of linear dependence into: "vectors are linearly independent if and only if there exists a zero linear combination of them, whose coefficients are not all zero." The definition of linear independence being the negation of this, it is therefore a pure problem of logic to reach the formal definition of linear independence. A pure problem of logic, but in a precise context, where the concepts make sense to the students from their intuitive background.

The evaluation of our research proved that students having followed an experimental teaching based on this approach are more efficient in the use of the definitions of linear dependence and independence, even in formal contexts.

Their scores for exercises such as the three questions quoted above are much higher than the scores of students having followed a more classical teaching (Dorier 1997).

Moreover, it is quite a discovery for the student to realize that a formal definition may be more practical than an "intuitive" one. Indeed, most of them keep seeing the fact that a vector is a linear combination of the others as a consequence of the definition of linear dependence. Therefore they believe that this consequence is the practical way of proving that vectors are or are not independent, even if that goes contrary to their use of these definitions.

This example is relevant with regard to the question about the role of formalism in linear algebra. Formalism is what students themselves confess to fear most in the theory of vector spaces. One solution is to avoid formalism as far as possible, or at least to make it appear as a final stage gradually. Because, from our historical analysis, we have pointed out evidence that formalism is essential in this theory, we give a different answer: formalism must be put forward in relation to intuitive approaches as the means of understanding the fundamental role of unification and generalization of the theory. This has to be an explicit goal of teaching. This is not incompatible with a gradual approach toward formalism, but it induces a different way of thinking out the previous stages. Formalism is not only the final stage in a gradual process in which objects become more and more general, it must appear as the only means of comprehending different aspects within the same language. The difficulty here is to give a functional aspect to formalism while approaching it more intuitively.

Linear dependence is a formal notion that unifies different types of dependencies, which interact with various previous intuitive conceptions. It has been shown above how in the historical development of linear algebra the understanding of this fact was essential for the construction of the concept of rank and partly of duality. In teaching, this questioning has to be made explicit, if we do not want misunderstandings to persist. Therefore even at the lowest levels of the theory the question of formalism has to be raised in interaction with various contexts where the students have built previous intuitive conceptions. The construction of a formal approach right from the beginning is a necessary condition for understanding the profound nature of the theory of vector spaces. In this sense, formalism has to be introduced as the answer to a problem that students are able to understand and to make their own, in relation to their previous knowledge infields where linear algebra is relevant. These include at least geometry and linear equations but may also include polynomials or functions, although in those fields one may encounter more difficulties.

Conclusion

The theory of vector spaces is a unifying and generalizing theory, in the sense that, historically, not only did it allow solving new problems[7] in mathematics, but it essentially unified tools, methods and results from various backgrounds in a very general approach. Thus its formalism is a constituent of its nature. Yet all the problems our students may solve with this theory could be solved with less sophisticated tools which they have already learnt (or at least are supposed to have learnt). Therefore the gains of this unification and generalization have to be understood by them, if we want them to accept this formalism and to use the theory correctly. General talk on the subject will not improve the situation, instead we have to build teaching sequences in which this idea can be understood in relation to a mathematical activity so that it becomes a personal reflection of the students. The students must be able to see the relationship between their knowledge and intuition in concrete contexts and the formal language of the theory of vector spaces. In our research work, we state that one must use this "meta-lever" to bring students to a personal understanding of the unifying and generalizing nature of the theory of vector spaces (see Dorier et al. 1995a and b; Dorier 1992, 1995b and 1997).

REFERENCES

Cramer, G. (1750): *Introduction à l'analyse des courbes algébriques*, Geneve: Crameret Philibert.

Dorier,J.-L. (1990a): *Contribution à l'éude de l'enseignement à l'université des premiers concepts d'algèbre linéaire. approcheshistorique et didactique,* Thèse de Doctorat de l'Université J. Fourier - Grenoble 1.

Dorier, J.-L. (1990b): *Analyse dans le suivi de productions d'étudiants de DEUG A en algèbrelinéaire,* Cahier DIDIREM n°6, IREM de Paris VII.

Dorier, J.-L. (1991): Surl'enseignement des concepts élémentaires d'algèbre linéaire à l'université, *Recherches en Didactique des Mathématiques* **11(2/3)**, 325-364.

Dorier, J.-L. (1992): *Illustrer l'Aspect Unificateur etGeneralisateur de l'Algebre Lineaire*, Cahier DIDIREM n°14,IREM de Paris VII.

Dorier J.-L. (1993):L'émergence du concept de rang dans l'étude des systèmes d'équations linéaires, *Cahiers du séminaire d'histoire desmathématiques* (2e série) **3** (1993), 159-190.

Dorier, J.-L., RobertA., Robinet J. and Rogalski M. (1994a): L'enseignement de

[7] These problems are far beyond the skills offirst year university students (see Dorier 1996b).

l'algèbre linéaire en DEUG première année, essai d'évaluation d'une ingénierie longue et questions, In *Vingt ans de Didactique des Mathématiques en France,* Artigue M. et al. (eds), Grenoble : La Pensée Sauvage,328-342.

Dorier, J.-L, Robert, A.,Robinet, J. and Rogalski, M. (1994b): The teaching of linear algebra in first year of French science university, in the *Proceedings of the 18th conference of the international group for the Psychology of Mathematics Education*, Lisbonne, 4 vol., 4:137- 144.

Dorier, J.-L. (1995a): Ageneral outline of the genesis of vector space theory, *HistoriaMathematica* **22(3)**, 227-261.

Dorier, J-L. (1995b): Metalevel in the teaching of unifying and generalizing concepts in mathematics,*Educational Studies in Mathematics* **29(2)**, 175-197.

Dorier J.-L. (ed.) (1997):*L'enseignement de l'algèbre linéaire en question,* Grenoble: LaPensée Sauvage.

Euler, L. (1750): Sur une contradictionapparente dans la doctrine des lignes courbes, *Mémoires del'Académie des Sciences de Berlin* **4**, 219-223, or, *Opera omnia*, 3 series - 57 vols., Lausanne: Teubner - Orell Fussli - Turicini,1911-76, 26:33-45.

Frobenius, G. F. (1875): Über das Pfaffsche Problem, *Journal für reine und angewandte Mathematik* **82**, 230-315.

Frobenius, G. F. (1879): Über homogene totale Differentialgleichungen, *Journal für reine und angewandte Mathematik* **86**, 1-19.

Ousman, R. (1996): *Contribution à l'enseignement de l'algèbre linéaire en première année d'université,* Thèse de doctoratde l'université de Rennes I.

Robert, A. and Robinet, J.(1989): *Quelques résultats sur l'apprentissage de l'algèbre linéaire en première année de DEUG*, Cahier de Didactique desMathématiques 53, IREM de Paris VII.

Rogalski, M. (1991): *Unenseignement de l'algèbre linéaire en DEUG A première année*,Cahier de Didactique des Mathématiques 53, IREM de Paris VII.

Rogalski, M. (1994):L'enseignement de l'algèbre linéaire en première année de DEUGA, *La Gazette des Mathématiciens* **60**, 39-62.

Vergnaud, G. (1990): La théorie des champs conceptuels, *Recherchesen Didactiques des Mathmatiques* **10(2/3)**, 133-170.