Seven Approaches to Solving a Question in Plane Geometry

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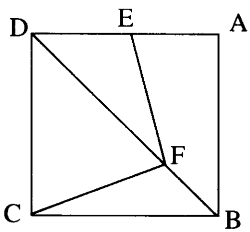
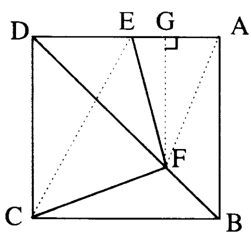
Question:

ABCD is a square of side a. E is the midpoint of DA. F is a point other than D on DB such that ∠EFC = 90°.

To show : DF = 3 FB

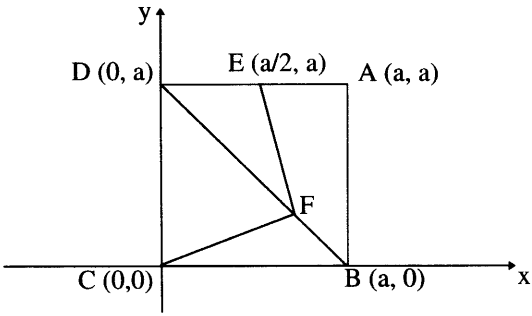


(I) Making use of the properties of cyclic quadrilaterals, symmetry and theorems in plane geometry



Observing that ∠EFC = ∠ADC = 90°, we can show that DEFC is a cyclic quadrilateral. Making use of the property of 'angles in the same segment' and the fact that ∠CDF = 45°, we can conclude that ∠CEF = 45°. Using the conditions for isosceles triangles, we can show that FC = FE. By joining FA and using symmetry, we can deduce that AF = FC. Hence AF = FE. Using the perpendicular bisector theorem, we can show that F lies on the perpendicular bisector of EA. Then the ratio of the lengths of DF and FB can be found easily by considering the lengths of DG and GA.

(II) Using coordinate geometry



Without loss of generality, let A, B, C and D be (a, a), (a, 0), (0, 0) and (0, a) respectively. Then E will be (a/2, a). We can make use of the fact that $\angle EFC = 90^\circ$ to find out the locus of F. This can be found by considering the slopes of EF and FC. Then we can get an equation relating the x-coordinate and y-coordinate of F. Hence, by simplifying, the ratio of the lengths of DF and FB could be obtained.

(III) Using vectors

Without loss of generality, let C be the origin, **CD** = **b** and **CB** = **c**. Then by putting **BF** = d **BD,** we want to show that d = 1/4. Obviously, we can express **CE** and **CF** in terms of **b** and **c**. Hence, we can express **FE** in terms of **b** and **c**. The condition that $\angle EFC = 90^\circ$ is equivalent to that the dot product of **FE** and **CF** is zero. After some simplications, d can be solved and hence the required ratio.

(IV) Using complex numbers

Without loss of generality, let the complex number 0 be represented by B in the Argand diagram. Let x, y and z be the complex numbers corresponding to C, E and F respectively, and let e be the length of FB. Obviously, we can find out x, y and z by considering their modulus and argument as follows:

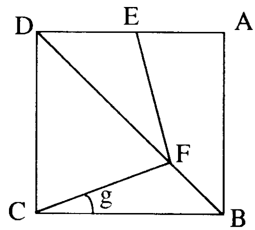
$$x = -a$$
$$y = -(a/2) + a\,i$$
$$z = e\,[\cos(3\pi/4) + i\,\sin(3\pi/4)]$$

As $\angle EFC = 90^\circ$, $CE^2 = EF^2 + CF^2$. Then we have

$$|y - x|^2 = |z - x|^2 + |z - y|^2$$

After some simplications, we can get an equation involving a and e. Solving this quadratic equation, we can express e in terms of a. Hence, we can find out the lengths of BF and FD and therefore their ratio.

(V) Using the properties of cyclic quadrilaterals and the sine rule



Let $\angle BCF = g$. We can get the following by using the properties of cyclic quadrilaterals:

$$\angle DCF = 90^\circ - g, \quad \angle DFC = 45^\circ + g, \quad \angle EFD = 45^\circ - g$$

and $\angle DEF = 90^\circ + g$.

Using the sine rule in ΔCDF, we get

$$\frac{DF}{\sin(90^\circ - g)} = \frac{CD}{\sin(45^\circ + g)} = \frac{CF}{\sin 45^\circ}$$

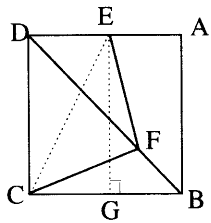
Similarly, from ΔBCF, we get $\dfrac{BF}{\sin g} = \dfrac{CF}{\sin 45^\circ}$

and from ΔDEF, we get $\dfrac{DE}{\sin(45^\circ - g)} = \dfrac{DF}{\sin(90^\circ + g)}$

From these three sets of equations, we can show that the ratio of DF to FB is cot g : 1. It suffices to show from these equations that $\cot g = 3$ or $\tan g = 1/3$.

(VI) Using the condition that FC = FE and the cosine rule

The condition that FC = FE can be proved by using the properties of cyclic quadrilaterals and the conditions for isosceles triangles.



By drawing the perpendicular from E onto CB (such that EG $\perp$ CB) and using the condition that FC = FE, we can find the length of EC in terms of a and hence the length of FC. Using the cosine rule for ΔCDF, we get

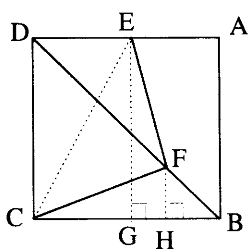
$$FC^2 = CD^2 + DF^2 - 2(CD)(DF)\cos 45^\circ$$

and similarly for ΔBCF, we get

$$FC^2 = BC^2 + FB^2 - 2(BC)(FB)\cos 45^\circ$$

Then, we can observe that DF and FB are the two roots of a quadratic equation. Solving this quadratic equation, we get DF = 3 FB.

(VII) Combining the proofs of (V) and (VI)



We can combine different parts of the proofs in (V) and (VI). The main difference is that we can find the exact measure of the angles concerned. Firstly, we can express FC in terms of a. Secondly, we can find the exact magnitude of $\angle$CEG. Hence we can find $\angle$FEG and $\angle$BCF. Then, based on the length of FC and by considering ΔHCF and ΔHBF, we can express FH and FB in terms of a. As we can easily compute the length of DB, the required ratio can be solved.