A Collective Reflection on the Transition from Secondary to University Mathematics through the Lens of the "Double Discontinuity" by Felix Klein

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Introduction

The German mathematician, Felix Klein (1908/1932, p. 1), observed that mathematics students face a "double discontinuity" as they move from secondary school to university, then back again to a career as schoolteachers:

1. The first 'discontinuity' concerns the well-known problems of transition which students face as they enter university, a main theme in research on university mathematics education (Gueudet, 2008).

2. The second 'discontinuity' concerns those who which return to school as teachers and the (difficult) transfer of academic knowledge gained at university to relevant knowledge for a teacher.

Indeed, Klein's notion of "double discontinuity" between university mathematics and secondary school mathematics has persisted in mathematics teacher education. Here in Hong Kong, concerns have been raised regarding teachers' experiences of learning mathematics in higher education and in secondary mathematics classrooms, including their identities, and mathematical knowledge constructed through those experiences. Many local teachers describe their contrasting experiences in these two contexts, confirming the "double discontinuity" phenomenon exists. To restate, Klein proposed that to address the school-to-university discontinuity:

a) taking the function concept as the focus of school instruction, and
b) making calculus the target of the secondary school curriculum.

Conversely, to address the university-to-school discontinuity, he proposed:

c) offering university courses that would show connections between problems in various fields of mathematics (e.g., algebra and number theory), and
d) developing university courses in elementary mathematics from an advanced standpoint (see Figure 1).

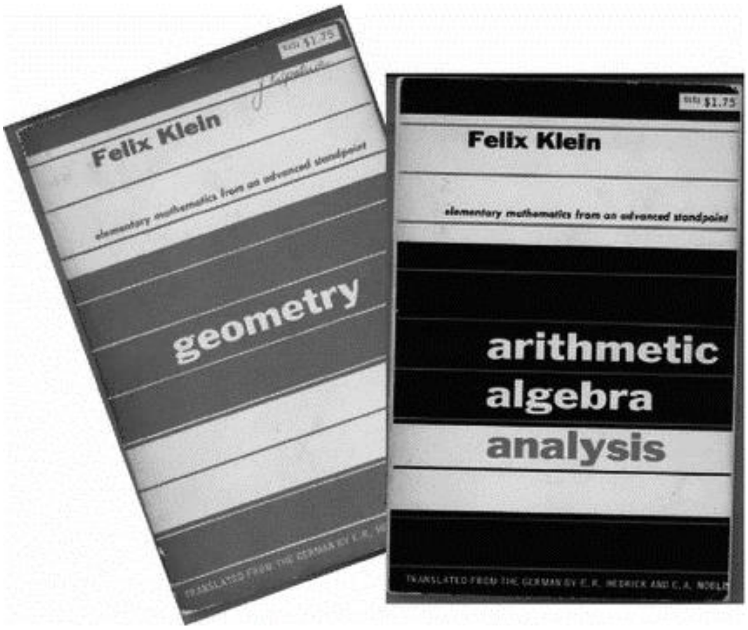

Figure 1

Finally, to address both discontinuities, Klein argued that instructors should make instruction livelier and more interesting, which meant that school mathematics should be more intuitive, less abstract, and less formal, and university mathematics should include more applied mathematics. Throughout his career, Klein saw school mathematics as demanding more dynamic teaching and consequently university mathematics as needing to help prospective teachers "stand above" their subject.

We are a group of mathematics prospective teachers (except for Oi-Lam and Biyao), who have been engaged in a course called "Contemporary Issues in Mathematics Education" (taught by Oi-Lam and Biyao; hereinafter referred as BMED 3091). In this final-year course, we as fifth-year B.Ed. students (with double-majors in mathematics and mathematics education) have been examining ways to support conceptual mathematics learning, particularly for transitioning from secondary to university mathematics more coherently and successfully. Through this experience, we realize how our own secondary–tertiary transition experiences have served as critical resources upon which we reflect to envision our future mathematics teaching. This learning opportunity allowed us to shed new light on mathematics (teacher) education to better support prospective teachers', and their future students', transitions between secondary school and university mathematics.

In this essay, we reflect on Felix Klein's comments through his lens of "double discontinuity", identify points of discontinuity from/to secondary and university mathematics, and propose some secondary mathematics lessons that may help bridge this discontinuity. We hope that by discussing the content of these lessons and why they are proposed, we can start a conversation about potential transformations that local secondary school curricula, university courses, and teacher education could undertake in better support teacher candidates and students of mathematics.

Reflections on Felix Klein's Double Discontinuities

Initial Reactions

Below are some of our initial feelings regarding the study of mathematics in university:

1. As stated by Laura on her choice of major, "If I have ever been told the contents of university mathematics, I might not have considered mathematics as a major."

2. The word used to describe the content is the same, i.e. “mathematics”, but how come we have the feelings that we are studying a totally different subject in secondary school and university?

3. Why do we have to learn that deep in mathematics? We will not be able to teach the same content to our students as these are not in the curriculum. It is a torture!

4. During our first year of study, we took three mathematics major courses: university calculus, linear algebra I, and modern mathematics which covered an introduction to logic, mathematical proofs and rigorous discussion on functions. In the first semester, most of our peers were already frightened about calculus as it involved much more theorems and topics than what was covered in the secondary mathematics curriculum. Also, the lecturer introduced these concepts in a symbolic, abstract, one-dimensional way instead of illustrating in various representations. Not to mention the concepts in linear algebra and the requirements for writing proofs, which was not covered at all in the M1 curriculum. Such huge differences between two curricula and two ways of delivery made us feel extremely lost and frustrated.

5. The one thing we feel good about studying mathematics in university is that we are able to feel how the students feel about studying mathematics. As Felix Klein stated about the first discontinuity, the curriculum bridge between secondary school and university mathematics is too weak. As a student, it is hard for me to relate secondary school curriculum to university content. Relating to the second discontinuity, what we learnt in university can hardly be applied in the secondary school curriculum. For example, in the strand of linear algebra, M2 only requires students to solve the equation with Reduced Row Echelon Form (RREF). While in university mathematics, it introduces a lot of terms like dimension, independence and orthogonal matrix. However, it does not fit in our syllabus. It does not allow more time for teacher to introduce more. Moreover, it doesn’t contribute any marks or rewards to student for learning extra curriculum.

In summary, it was truly difficult for us to convert our knowledge gained in university into educational form of school mathematics. As a consequence of both discontinuities, teachers may lose sight of academic mathematics after university studies and, thus, teach on the basis of experiences from their own schooldays. We once had the belief that we were not as ‘familiar’ to school (HKDSE) mathematics as we were in Secondary 6, where we ‘drill’ on the content and exercises everyday back in the past. One of the major reasons contributed to this phenomenon is on how we received training during our tertiary education: we separate ‘mathematics education’ to ‘mathematics + education’. Not only the prospective teachers may not be able to combine what they have learnt from both departments, we often have such questions like: ‘Why are we learning such high level of mathematics, such as real analysis, complex analysis, and abstract algebra, where they are useless in our daily teaching?’ In fact, in addition to familiarizing future mathematics teachers with some critical thinking, these advanced mathematics studies at the university will also play a key role in answering students’ questions in later teaching. That is why Klein proposed that we, mathematics educators, should view school mathematics at a higher standpoint and we believe that both school curriculum and university programmes need to make amendments in order to achieve this goal.

Experiencing the Double Discontinuities as Prospective Teachers

As prospective mathematics teachers, what Klein mentioned about the “double discontinuity” is indeed one of the words that we could use to describe our five years of university studies. We remembered during our first year of studies, the MATH courses alone already took up most of our study time, because they required much effort than the other courses. In particular, being introduced to the rigorous mathematics really did shock our eyes and it was as if there were no room for mistakes. It took us almost a year and a half before we could get used to university mathematics.

But when talking about a discontinuity from secondary school to university, we believe it would be the lack of elements of mathematical modelling in secondary schools. During our third year of study, we have the chance to take the course “Mathematical Modelling”. In that course, topics such as linear regression (or in secondary school’s term, best-fit line) were introduced. And immediately we thought to ourselves, could secondary school students also grasp the main idea of linear regression (and therefore, other mathematical modelling methods)? Would it be nice for secondary students to be exposed to this kind of ideas so that they could have better transition to university mathematics?

Another discontinuity from our own experiences was that, when we transitioned to teaching secondary mathematics during our practicum, we often found it uneasy to explain some abstract mathematical, or logical concepts with limited terminologies or mathematical knowledge that students knew. In particular, many students might find questions involving proof / disproof confusing as they had not understood the rationale behind deductive and inductive logic. Indeed, as proposed by Klein, when mathematics education graduates return to secondary schools, they may find that the mathematical concepts are “discontinued” from their university experience.

However, the above claim did not fully apply to Alvin’s first teaching practicum. In his first teaching practice, which he conducted at his alma mater, he was allowed to introduce some of the mathematical concepts he learned at university in his teaching. As such, he has had the chance to introduce a group presentation activity to his secondary school class, where the students would form groups and present some mathematical concepts at the university level (but without rigorous computations and proofs, nor abstract concepts). Still, that could be of a special case only, as not all schools have the “luxury” to do so. Nevertheless, we believe what Alvin did was beneficial for students to see mathematics beyond the school curriculum, and we recommend schools offer this kind of opportunity so as to broaden students’ horizons.

On the Teaching and Learning of Functions and Calculus

For the school-to-university discontinuity, we agree on Felix Klein’s suggestions generally. Indeed, the local secondary mathematics curriculum does not emphasize the concepts of functions. Currently, the idea of functions is introduced in a procedural way instead, resulting in students’ (and prospective teachers') weak understanding of functions. We recall an interesting finding about a research paper that we learned from one of our peers last year, which revealed an incomplete conception or misconception held by both US and Korean mathematics teachers on the topic of functions (Yoon & Thompson, 2020). Here in Hong Kong, we believe that most students have similar misconceptions. As the concept of functions is foundational to higher mathematics, such as calculus, analysis, linear algebra and even abstract algebra, students who do not show mastery on the learning of function would find it difficult to understand higher mathematics conceptually. If more emphasis could be put on the concept of functions in the secondary curriculum, contents of higher mathematics might become more accessible to students, thus alleviating the school-to-university discontinuity.

In Hong Kong, many students found university mathematics, i.e. calculus, hard to pick up from their foundational knowledge in their secondary schooling. As argued by Klein, students do not have sufficient exploration in school mathematics on calculus field, leading to disastrous grades in mathematics courses in university. Therefore, he proposed that calculus should be the target in the secondary curriculum. The effect of such an arrangement on tackling the discontinuity is self-explanatory, as students feel less disjoint about the university calculus courses if more contents about calculus have been explored in secondary school. Currently, the core mathematics curriculum does *not* involve calculus and it is designed for reasons. One of the reasons is that calculus may not be as significant as other topics put in current compulsory mathematics curriculum in terms of developing basic mathematics competence and mathematical literacy. There is only limited amount of time for mathematics education and most of the students do not require the study of higher mathematics in their further study. Targeting calculus in the secondary school curriculum may narrow the gap between secondary and university mathematics. However, for those who do not study university mathematics, will they suffer?

We do recognize Felix Klein's suggestions, yet we also believe curriculum developers should strike a balance between tackling school-to-university discontinuity and the development of basic mathematics competence and mathematical literacy. A possible way of doing so is to encourage students to study M1 or M2 by increasing their significance in admission for majors that require the study of university mathematics. As calculus is one of the main targets in the curricula of both M1 and M2 while students can choose to study extended mathematics or not, such a measure may address the school-to-university discontinuity for students, who will pursue higher mathematics in undergraduate studies, while maintaining the development of basic mathematics competence and mathematical literacy for students who do not need higher mathematics.

Addressing the Double Discontinuities

Reconstructing Elementary Mathematics From a Higher Standpoint

Felix Klein observed the double discontinuity and proposed some directions of resolving / addressing the issue. We see what he meant by proposing undergraduate courses that connect different fields as well as discussing elementary mathematics from a higher standpoint. From our experience, the more advanced mathematics that we study, the more disconnected we feel about different fields of mathematics. This may be caused by the advanced, abstract content of each mathematics course, resulting in absence of time for the lecturer to connect the content knowledge in a field to another. If we learn higher mathematics in such a disjoint way, the relationships among ideas from different fields may become unclear, thus restricting us to connect and apply these ideas back to elementary mathematics (university-to-school discontinuity). Therefore, providing courses that connect different fields could help us see the big picture, understand their relationships, lead to more comprehensive understanding on mathematics and boost higher capability of transferring our learning of university mathematics back into the teaching of elementary mathematics.

Apart from that, discussing elementary mathematics from higher standpoint is another crucial, perhaps the most important, factor of tackling university-to-school discontinuity. It is true that showing the connections of mathematics from different areas could help us comprehend mathematics in a deeper way. Yet, it is us, prospective teachers, who will make the connection between elementary and university mathematics, and this could be challenging for us to do without assistance from others. If we are able to situate ourselves “from a higher standpoint”, as Klein would say, we would see that elementary mathematics is a small part of the big picture called higher mathematics. Likewise, Tom believes that courses which discuss elementary mathematics from a higher standpoint, such as BMED3091, has guided him to revisit and redevelop his conceptual understanding on elementary mathematics from the perspective of higher mathematics. Overall, we learned to tackle secondary mathematics topics (e.g., the idea of amount of change and generalizing angle measures) in a more logical and deeper way, which we had not experienced before. This was exactly what Klein meant by the continuity from higher to elementary mathematics. When participating in BMED3091, most of us were inspired by the lessons and had our conceptions reshaped and completed. So, we are deeply convinced by Felix Klein’s comments, especially about addressing the university-to-school discontinuity by discussing elementary mathematics from a higher standpoint. And therefore, university should consider designing and offering prospective teachers this kind of courses so that they can reconstruct their mathematical understandings and be more prepared to transforming school teaching.

Redesigning Lessons on Particular Mathematical Topics

Data handling. We may address the discontinuity that occurs in the Data Handling strand with a lesson on introducing the essence of best-fit line in a lower form mathematics classroom. In particular, we could focus on the topic named “Use and Misuse of Statistics", where the nature of population and sample are introduced. Within this topic, it seems unnatural to most secondary students, at first glance, how the sample size affects the statistical result. Therefore, we could adopt the idea of best-fit line here, where students would be divided into groups, and based on different sample sizes, draw a different best-fit line. By comparing the groups’ work in front of the whole class, they can directly observe the “accuracy” of the prediction made by different sample size. That is, the larger the sample size, the more accurate the statistical result can be. The main idea here is not to introduce additional topics to the current curriculum, as that would be difficult to implement in current Hong Kong secondary mathematics education context. However, we believe that teachers could make use of some concepts they have learned in university and indirectly adopt these concepts in their teaching.

Algebra. Regarding algebra, we could propose a secondary mathematics lesson related to the topic of simultaneous equations (learnt in secondary school) and the theory of simultaneous linear equations (learnt in university). This idea originates from Marco, who identifies a connection between these topics. That is, students learn how to solve simultaneous equations by two methods (the method of substitution and the method of elimination) in the junior form, while students in university also learn how to solve simultaneous equations by the reduced row echelon form (RREF). Hence, if the concept of reduced row echelon form was introduced in the secondary school level under a relevant topic, i.e. solving equation in two unknowns, students could build a good foundation when they learn it in a more advanced way in university. The idea is to let students have an initial exploration towards an advanced topic so that they will not be scared by the big difference between secondary and university mathematics when first transitioning into university mathematics lectures. At the same time, schoolteachers could teach what they have learnt in university in their secondary mathematics lessons. In this way, their knowledge gained from university will not remain only in their memory of university mathematics, but they can also apply them by teaching in secondary schools.

Logic and its applications. Apart from offering university courses that would show connections among problems in various fields of mathematics (e.g., algebra and number theory; algebra and linear algebra), we propose that we should amend the secondary school curriculum, in particular, adding logic as a part of the syllabus. As Thomas suggests, teachers could teach basic logic ideas, from decomposing statements, to using logical operators to show the ideas of common language in mathematics (e.g. ‘if ..., then ...’, ‘if and only if’). For example, if students understand the meaning of negation of a statement, they would connect to topics such as why we could provide counterexamples to disprove some statements, which is exactly the relationship between ‘for all (∀)’ and ‘there exists (∃)’. Teachers could connect all these logics back to the original mathematics syllabus so that students would have an easier time to read mathematical texts, hence alleviating the gap between secondary school and university mathematics.

Similarly, we think it would be great to introduce the logic flow for secondary school students. There are reasons for every mathematical step; so, we need to justify and make sense of every step. Hence, learning the logic flow for students is important. If we had learnt the truth table in secondary school mathematics, we would be able to distinguish which condition is possible to happen and which condition will never happen. For example, suppose that we know two angles of a triangle are 70° and 90 ° respectively, and that the base angles of an isosceles triangle are always equivalent. Then, we can deduce that the triangle cannot be an isosceles triangle, since the two conditions cannot be satisfied simultaneously. For these reasons, Elaine thinks that the introduction of logic is important for secondary school students. To introduce the idea of logic, we can first use daily life examples. For instance, consider the statement, "we take medicine only we are sick". Hence, if we take medicine, then it implies that we are sick. Translating into mathematics, we can apply the same logic to properties of isosceles triangles. If we have an isosceles triangle, then it will satisfy all the properties of the isosceles triangle. To test for their understanding, we can ask the students to answer some questions with the same logic and then create a situation that fulfils a certain logic flow.

Closing Remarks

Addressing the discontinuity of transition from secondary school to university mathematics is a common challenge in educational systems worldwide. Relevant literature suggests a significant need in confronting not only different environments and modes of teaching and learning, but more importantly, the change from a computational to a proof-based view of mathematics (Gueudet, 2008). In Hong Kong, the culture of exam-oriented procedural knowledge perpetrates secondary mathematics education, while in university mathematics courses, students are introduced to reasoning and proving through increased emphasis on the precision and rigor of the mathematical language, which is a new approach for students entering university-level mathematics learning (Liang, Ng,

& Chan, under review). As prospective mathematics teachers and teacher educators, we resonate very much with the phenomenon of "double discontinuity" and Felix Klein's call for teaching "elementary mathematics from an advanced standpoint".

Furthermore, Oi-Lam and Biyao have become interested in understanding how prospective and in-service mathematics teachers' experiences in the context of transitioning from school and university classrooms as learners. In particular, we underscore the importance of these learning experiences for knowledge and identity construction as mathematics teachers. We advocate that it is important to understand one's own experience, as well as how it is constructive in shaping one's values and identities, in this case, as associated with school mathematics teaching. Oi-Lam and Biyao's recent research showed that the prospective teachers also formed coherent (as opposed to "discontinued") connections between contexts in many aspects. For example, they valued the role of advanced concepts and abstract notations in high school curricula but were mindful about situating these ideas in contexts accessible to students (Liang, Ng, & Chan, under review). Likewise, in Thomas and Elaine's reflections, they intended to teach beyond procedures and computation by focusing on logics and reasoning. Thus, we also invite the readers to pause and reflect on their own experience as mathematics learners, upon which, to consider how they may reorient their practice, as well as play a role in catalyzing possible changes in the secondary and university mathematics curricula in order to address the century-long "double discontinuity".

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