A Theorem on Intersecting Chords on a Circle

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Introduction

Starting from a locus problem about a point and a circle, a nice theorem on intersecting chords is found. The theorem is not commonly known to teachers and students, but the statement could be easily understood. A proof of the theorem is presented using Ceva's Theorem and similar triangles.

The Locus Problem

Let P be a fixed point outside a fixed circle. Two lines from P cut the circle at four points A, B, C and D. Let Q be the intersection of the diagonals of the quadrilateral formed by the four points (Figure 1). As the two chords move, find the locus of Q.

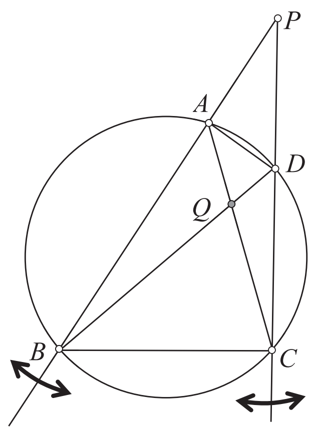

Figure 1 – Locus of Q as two chords move

Readers are suggested to try an exploration to obtain the locus of Q and test your finding.

Answer. Using the trace function of a dynamic geometry software, we find that the locus is a chord on the circle. Where are the endpoints of the chord? If we consider the limiting case when the chord *PCD* moves away and becomes a tangent, then *C* and *D* coincide at the point of contact of the tangent and the “diagonals” *AC* and *BD* intersect at the same point, i.e. $C = D = Q$. By symmetry, the other endpoint must be the other point of contact from *P*. Hence we conclude that the locus of *Q* is the chord of contact from *P* to the circle.

The Theorem

We can formulate the answer to the locus problem into the following theorem about chords on a circle.

Theorem.
Let *P* be a point outside a circle. Two lines from *P* cut the circle at four points to form a quadrilateral *ABCD*. Then the chord of contact of *P* to the circle passes through the intersection point of the diagonals *AC* and *BD* (Figure 3).

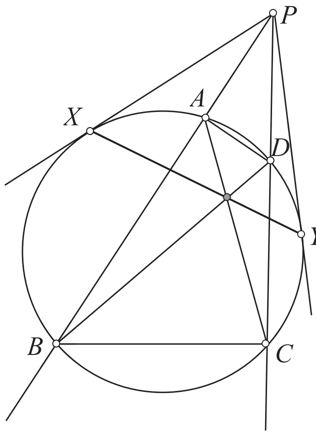

Figure 3

Proof.
To prove the theorem, we denote the contact points of the two tangents from *P* to the circle by *X* and *Y* (Figure 3). It suffices to show that *XY*, *AC* and *BD* are

concurrent. As these are the three main diagonals of a cyclic hexagon, we may apply the following lemma to check the concurrency.

Lemma.

Let *ABCDEF* be a cyclic hexagon. The three main diagonals *AD*, *BE* and *CF* are concurrent if and only if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

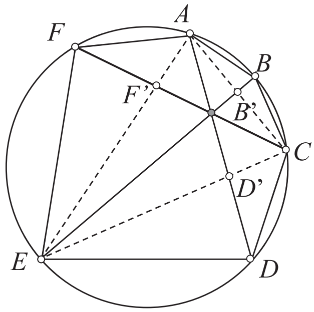

Figure 4

Proof of Lemma.

We use Ceva's Theorem to prove this. Consider $\triangle ACE$, and let *AD*, *BE* and *CF* cut the sides of the triangles at *D*', *B*' and *F*' respectively (Figure 4). By the theorem of Ceva, *AD*, *BE* and *CF* are concurrent if and only if

$$\frac{AB'}{B'C}\cdot\frac{CD'}{D'E}\cdot\frac{EF'}{F'A}=1$$

But

$$\frac{AB'}{B'C}=\frac{[BAE]}{[BCE]}=\frac{\frac{1}{2}AE\cdot AB\sin\angle BAE}{\frac{1}{2}EC\cdot BC\sin\angle BCE}=\frac{AB}{BC}\cdot\frac{AE}{EC}$$

Here $[P_1P_2P_3]$ represents the area of $\triangle P_1P_2P_3$; and the last equality holds as $\angle BAE + \angle BCE = 180^{\circ}$ for cyclic quadrilateral *ABCE*.

Similarly, we have $\frac{CD'}{D'E}=\frac{CD}{DE}\cdot\frac{CA}{AE}$ and $\frac{EF'}{F'A}=\frac{EF}{FA}\cdot\frac{EC}{CA}$.

Hence

$$\frac{AB'}{B'C}\cdot\frac{CD'}{D'E}\cdot\frac{EF'}{F'A}=\left(\frac{AB}{BC}\cdot\frac{AE}{EC}\right)\cdot\left(\frac{CD}{DE}\cdot\frac{CA}{AE}\right)\cdot\left(\frac{EF}{FA}\cdot\frac{EC}{CA}\right)$$

$$=\frac{AB}{BC}\cdot\frac{CD}{DE}\cdot\frac{EF}{FA}\cdot\frac{AE}{EC}\cdot\frac{CA}{AE}\cdot\frac{EC}{CA}$$

$$=\frac{AB}{BC}\cdot\frac{CD}{DE}\cdot\frac{EF}{FA}$$

Hence the necessary and sufficient condition for the three main diagonals to be concurrent is $\frac{AB}{BC}\cdot\frac{CD}{DE}\cdot\frac{EF}{FA}=1$, i.e. $AB\cdot CD\cdot EF = BC\cdot DE\cdot FA$.

This completes the proof of the lemma.

For the proof of the theorem, we need to check the condition in the lemma that $XA\cdot DY\cdot CB = AD\cdot YC\cdot BX$. From the tangents PX and PY and property of angles in the alternate segment, we have $\triangle PXA \sim \triangle PBX$ and $\triangle PDY \sim \triangle PYC$.

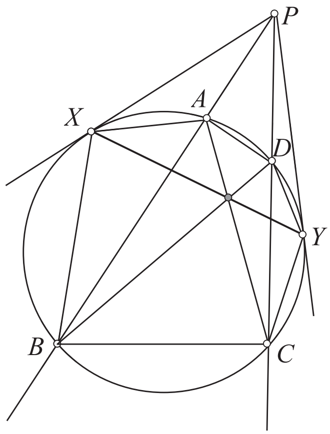

Figure 5

Hence

$$XA = BX\cdot\frac{PX}{PB} \qquad \text{and} \qquad DY = YC\cdot\frac{PD}{PY} \qquad \text{....(1)}$$

Consider another pair of similar triangles $\triangle PCB \sim \triangle PAD$, we have

$$CB = AD \cdot \frac{PB}{PD} \qquad \text{….(2)}$$

Putting (1) , (2) and $PX = PY$ (tangent property) together, we have

$$XA \cdot DY \cdot CB = BX \cdot YC \cdot AD \cdot \frac{PX}{PB} \cdot \frac{PD}{PY} \cdot \frac{PB}{PD} = BX \cdot YC \cdot AD.$$

This completes the proof of the theorem.

Generalizations

From Inside to Outside

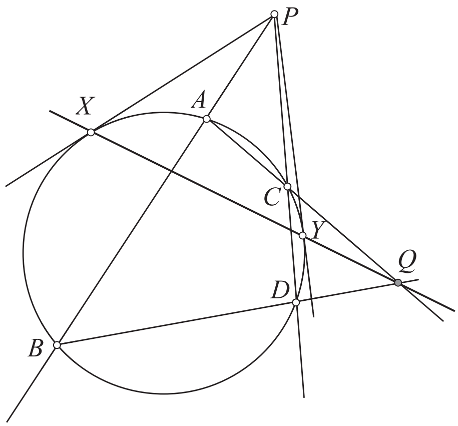

Figure 6

The theorem holds also for the other configuration of *C* and *D* (Figure 6). The intersection point of *AC* and *BD* lies outside the circle is also lying on the chord of contact (extended). In other words, *AC*, *BD* and *XY* are concurrent. In circle geometry, the point *P* and the chord of contact *XY* are called the pole and polar with respect to the circle, which have many fascinating properties.

We start with a point *P* outside a given circle and two chords are drawn to the circle which form a quadrilateral. Then both the intersection point of the diagonals and the intersection point of the opposite sides lie on the chord of contact. This provides a straight-edge-only construction of the chord of contact from a given point outside a circle.

From Circle to Conics

The theorem holds not just for circles, but in general for all conic sections. One way to see this is by considering central projections. Central projections send a circle on a plane to another conic section on another plane, and send lines to lines while retaining collinearity, concurrency and tangency (Figure 7).

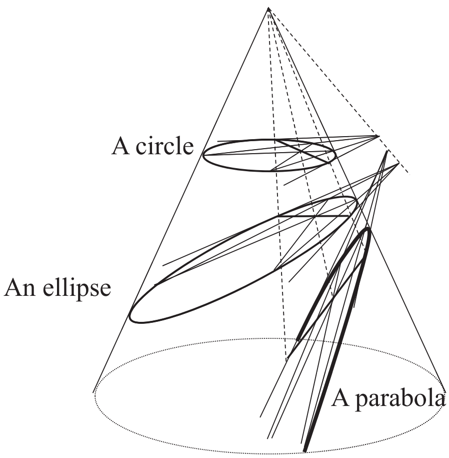

Figure 7: Central projections sending a circle to an ellipse and a parabola

Conclusion

I came across the theorem as a lemma of a harder geometric problem. The proof of this requires the use of the Pascal's Theorem for cyclic hexagon. I would like to find a proof of this without using this harder theorem. The theorem is fascinating, as it could be easily stated in school geometry, yet this property of concurrent chords is not well-known to us. I would like to see whether there is a simpler proof for secondary school students.

References

Coexeter, H.S.M & Greitzer S.L. (1967). *Geometry revisited.* Washington DC, USA: Mathematical Association of America.

Honsberger, Ross (1995). *Episodes in nineteenth and twenthieth century Euclidean geometry*. Washington DC, USA: Mathematical Association of America.

Poon, W. H. B. (2007). Several proofs of Ceva's Theorem by students. *EduMath*, *25,* 76-82.

Yaglom, I., & Shenitzer, A. (1973). *Geometric transformations III: Affine and projective transformations.* Washington DC, USA: Mathematical Association of America.

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