CONCAVE UP EVENTUALLY GREATER THAN CONCAVE DOWN?

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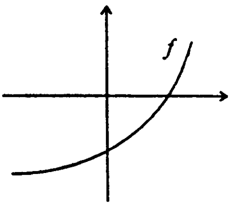
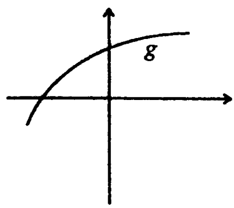
Let f be a function that is increasing and concave up, and g be a function that is increasing and concave down. Graphically, we know that f and g are as follows.





Since f increases faster than g, we may intuitively think that f will eventually be greater than g. The main question of this short note is : Must there exist a point x_1 such that $f(x_1) > g(x_1)$?

Consider the functions $f(x) = x - ln\,(x)$ and $g(x) = x + ln\,(x)$, where $x > 1$. We clearly have $f'(x) = 1 - \frac{1}{x}$, $f''(x) = \frac{1}{x^2}$, i.e. f is increasing and concave up, and $g'(x) = 1 + \frac{1}{x}$, $g''(x) = -\frac{1}{x^2}$, i.e. g is increasing and concave down. But we always have $f(x) < g(x)$. In other words, our intuition is not correct. Although f increases faster than g, the difference is the rates may not be large enough for f to overtake g.

Now let us add another condition. Suppose that there exists a point x_0 such that $f'(x_0) > g'(x_0)$. Does the point x_1 hypothesized above exist? We will show that now it does.

To simplify our proof, let $h = f - g$. Then $h'(x_0) > 0$ and $h''(x) > 0$ for all x.

Consider any $x_1 > x_0 - \dfrac{h(x_0)}{h'(x_0)}$. By Taylor series, we have

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\begin{aligned}
h(x_1) &= h(x_0) + h'(x_0)(x_1 - x_0) + \frac{1}{2}h''(c)(x_1 - x_0)^2, \text{ where } c \in (x_0, x_1) \\
&\geq h(x_0) + h'(x_0)(x_1 - x_0) \\
&> h(x_0) + h'(x_0)\left(x_0 - \frac{h(x_0)}{h'(x_0)} - x_0\right) \\
&= 0, \text{ which concludes the proof.}
\end{aligned}
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