Learning Mathematics through Inventing in Classroom

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The current mathematics education reform presents a vision for school mathematics which, among other important ideas, emphasizes students' conceptual understanding, reasoning, and problem solving. However, all too often students hold the misconception that there is only one "right" way to approach and solve a problem; and learning math is mostly memorizing rather than understanding. Therefore, although many students appear to know mathematical algorithms or procedures, they fail to correctly apply the knowledge to problem situations because of their lack of conceptual understanding. Conceptual understanding implies not only knowing the mathematics, but also knowing when and how to use the knowledge solving novel problems. I have argued elsewhere that in order for students to conceptually understand mathematical algorithms, we should encourage them to invent their own algorithms in problem situations (Cai, Moyer, & Laughlin, in press). In this article, I argued that we should provide students opportunities to learn mathematics through their own inventing in classroom.

The idea that students should be provided opportunities to learn mathematics through their own inventing is not new. In fact, a half century ago, Piaget (1948/1973) had already indicated that to understand is to invent. "To understand is to discover, or reconstruct by rediscovery, and such conditions must be complied with if in the future individuals are to be formed who are capable of production and creativity and not simply repetition" (Piaget, 1973, p. 20). Recently, National Council of Teachers of Mathematics (in the U.S.) suggests that school mathematics should increase attention on creating algorithms and procedures and decrease attention on memorizing rules, procedures, and algorithms (NCTM, 1989, 1991). In particular, NCTM indicates that "[m]athematics is learned when learners engage in their own invention and impose their own sense of investigation and structure" (1991, p. 144).

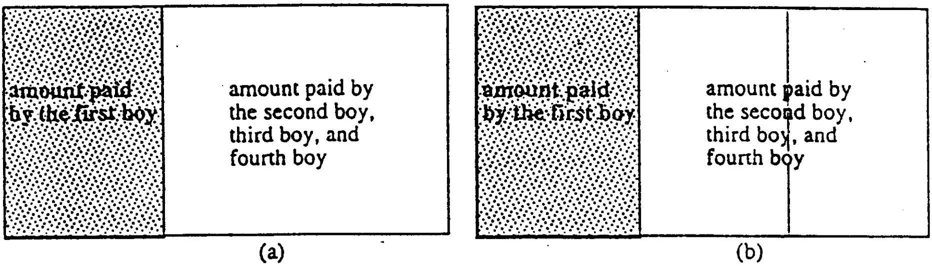
Learning is a process of knowledge construction. In order for our students to have conceptual understanding of mathematics, they need to experience a constructive process, which is based on students' own thinking. Mathematical knowledge is not directly recorded or absorbed but is constructed by each individual learner. In learning through inventing, students actively participate in processes of knowledge construction and make sense of mathematics. They become active participants in creating knowledge rather than passive receivers of rules and procedures. As Steen (1986) indicated: "the mathematics curriculum must not give the impression that mathematical and quantitative ideas are the product of authority or wizardry" (p. 6). Students have "authority" over the knowledge through their own inventing and constructive processes.

Are Students Capable of Inventing?

Many researchers (e.g., Cai, Magone, Wang, & Lane, 1996; Carpenter et al., 1989; Kamii, 1994; Resnick, 1989) have investigated students' mathematical thinking and indicated that even young children can invent their own mathematics. For example, when a first-grade student was asked to find the sum of 60 and 35, he did not use the traditionally taught addition algorithm. Instead, he used an algorithm he had invented himself: first add the tens then the ones. He explained: "I would take away the 5 from the 35. Then I'd add the 60 and the 30, which equals 90. Then I'd bring back the 5 and put it on the 90, and it equals 95" (Resnick, 1986, p. 165).

In another example, fourth- and fifth-grade students were asked to solve the following problem: "*Four boys bought a boat for $60. The first boy paid one half of the sum of the amount paid by the other boys; the second boy paid one third of the sum of the amount paid by the other boys; and the third boy paid one fourth of the sum of the amount paid by the other boys. How much did each boy pay ? Show how you find your answer.*" At a first glance, this problem seems to be too difficult for fourth- and fifth-grade students to solve. In fact, normally junior high school students will set up a system of four equations to solve this problem. However, with a careful guidance of teachers, even fourth- and fifth-grade students were able to invent a solution to this problem. Students drew a rectangle to represent the amount of money paid by four boys. As shown in Figure 1 (a), let the shaded region represent the amount paid by the first boy and unshaded region represent the amount paid by the remaining three boys. Since the first boy paid one half of the sum of the amount paid by the other boys, the shaded region is equal to one half of the unshaded region. Then the unshaded region was cut into two equal pieces, as shown in Figure 1 (b). Each of the three regions represents one third of the amount paid by the four boys. Therefore, the amount paid by the first boy is equal to one third of the amount paid by the four boys, which is \$60. That is, the first boy paid \$20 ($= \frac{1}{3} \times \$60$).

Figure 1. A Visual Solution



Obviously, not all students were able to invent and understand this visual approach. Teachers could ask those students who "invented" this approach to present their solutions. Then teachers could ask students to figure out the amount paid by the second boy using a similar approach, shown in Figure 1.

The third example is that a group of sixth-grade students were asked to solve the following problem: "The average of Ed's ten test scores is 87. The teacher throws out the top and bottom scores, which are 55 and 95. What is the average of the remaining set of eight scores ?" Students were asked to explain their solution process. Commonly, students used some variant of the average algorithm to solve the problem. In the explanations of their solution processes, they either used mathematical expressions or written words. For example, one student explained in words that: "First I multiplied 87 × 10 and got 870. Then I subtracted 55 from 870 and got 815. Then I subtracted 95 and got 720. Then I divided 720 by 8 and got 90." Another student explained symbolically that: $87 \times 10 - (55 + 95) = 720$. $720 \div 8 = 90$.

Some students used a unique strategy to solve the problem. Using properties of average, the student determined that the average for the remaining eight scores must be between 55 and 95 since an average is always between two extreme values. Then the student drew 10 circles and put 95 on the first and 55 on the last, leaving eight empty circles. Using these two numbers, the average would be 15 [$(95 + 55) \div 10 = 15$]. Then the student said that each of the ten circles should get 15. But 15 is

72 less than 87 (the average for 10 scores). He then multiplied 72 by 10 and got 720. 720 ÷ 8 = 90. Thus 90 became the average of the remaining eight scores after the top and bottom scores were thrown away. In this solution, the student viewed throwing away the top and bottom scores as taking 15 away from each circle. By inventing this approach, which works for any problem like this one, these students demonstrated an incredible understanding of averaging (Cai, Moyer, & Grochouski, 1997).

These examples not only suggest that students are capable of inventing their own mathematics, but also suggest that students displayed in-depth understanding of the relevant concepts. Creating a classroom environment that emphasizes learning through mathematical exploration and inventing is a challenging, but important goal for classroom mathematics teachers.

What Is the Role of Teachers in Learning through Inventing ?

Learning mathematics through inventing is not only consistent with the vision of current mathematics education reform (NCTM, 1989), but also consistent with both the constructivist and sociocultural perspectives of learning mathematics (Cobb, 1994). According to the constructivist perspective of learning mathematics, learning is an individual process and each individual responds to learning situations in terms of the meaning he/she has for them (Cobb, 1994). Therefore, the teachers' role is to provide appropriate environments and design "good" problems that facilitate students to invent and construct their own knowledge. "Good" problems engage students in mathematical exploration and invite students to learn through their invention. A "good" mathematical problem should have at least one of the following features:

(1) It has the potential to create a safe and productive learning environment in which students are inclined to express their thinking about mathematical structures and relationships;
(2) It is embedded in familiar and friendly context and has the potential to lead students into unfamiliar and important mathematical territory and, in particular, to lead students into territory that relates to the curricular agenda.
(3) It is truly problematic; and
(4) It allows for exploration from various perspectives.

Existing "good" problems are far less than what we need for classroom instruction. Collaborative efforts between mathematics teachers and mathematics education professors are desperately needed to create "good" mathematical problems which are suitable for classroom use.. Listed below are several sample problems teachers may want to use in their classrooms. These problems were adopted from various publications (e.g., Cai et al., in press; Cai, Jakabcsin, & Lane, 1996; Heid, 1995; Zhang, Sawada, & Becker, 1993). Hope these problems will serve as a catalyst for mathematics teachers to develop their own "good" problems.

Sample Problem 1

Offer 1: At Timmy's Tacos you will earn $4.50 an hour. However, you will be required to purchase a uniform for $45.00. You will be expect to work 20 hours each week.

Offer 2: At Kelly's Car Wash you will earn $3.50 an hour. No special uniform is required. You will be expect to work 20 hours each week.

Which offer will you take? Justify your answer.

Sample Problem 2

Eight adults and two children need to cross a river. A small boat is available that can hold one adult, or one or two children. Everyone can row the boat. How many one-way trips does it take for them to all cross the river ? Show or explain how you got your answer.

Sample Problem 3

There are three parks, A, B, and C in which many boys and girls are playing. The area of Park A is 500 square yards. The area of Park B is also 500 square yards. The area of Park C is 300 square yards. There are 40 children playing at Park A. There are 30 children playing at Park B. There are also 40 children playing at Park C. Which Park is the most crowded ? Explain your answer.

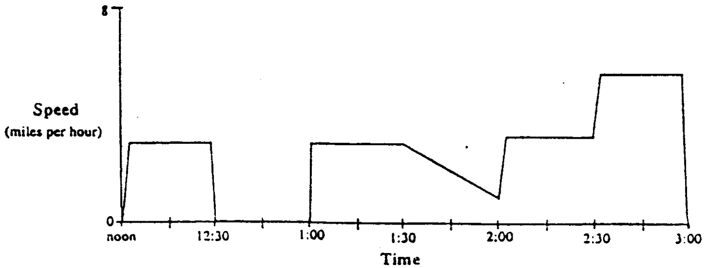
Sample Problem 4

There is a rectangular field. You are asked to design a small garden so that the area of the garden is equal to half of the area of the field. What is your design ? Show your design.

Sample Problem 5

Use the following information and the graph to write a story about Tony's walk.

At noon, Tony started walking to his grandmother's house. He arrived at her house at 3:00. The graph below shows Tony's speed in miles per hour throughout his walk.



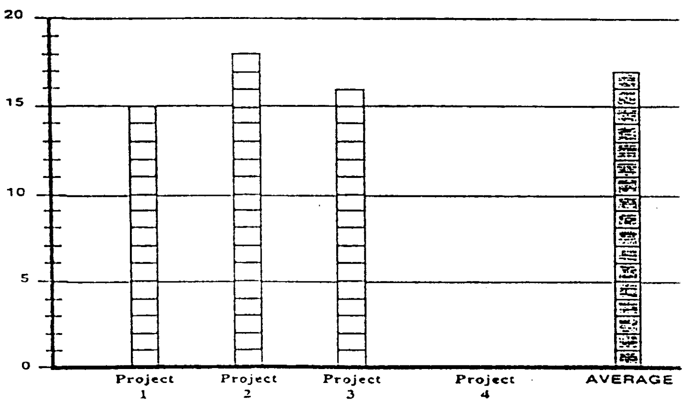
Write a story about Tony's walk. In your story, describe what Tony might have been doing at the different times.

Sample Problem 6

The ratio of females to males in a school band is 7 to 4. If three females and twelve males are absent from practice, the ratio of females to males is 5 to 2. How many members of the band attend practice ? Show how you found your answer.

Sample Problem 7

Anita has four 20-point projects for her class. Anita's scores on the first 3 projects are shown below.



What score must Anita get on Project 4 so that her average for the four projects is 17 ? How did you find your answer ?

According to the social perspective of learning mathematics, on the other hand, learners become acculturated by participating in cultural practices. Learning is a social process in which each individual learns mathematics through social interaction, meaning negotiation, and reaching shared understanding. For socioculturalists, the teacher's role is to mediate between students' personal meanings and the culturally established mathematical meanings of wider society. From this point of view, one of the teacher's primary responsibilities is to appropriate students' inventions into a wider system of mathematical practices (Cobb, 1994). The teacher's mediating role is evident in the following example.

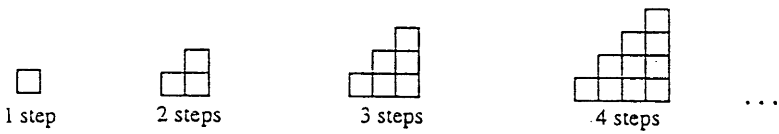
The teacher was guiding a group of 6th grade students solving The Staircase Problem shown in Figure 2. Students were asked to find their answer in as many different ways as they could. Students worked on this problem in pairs. After 20 minutes, each pair had at least one way to find out the number of blocks needed to build a staircase of 9 steps. Then each pair was asked to present their solutions for the number of blocks needed to build a staircase of 9 steps. Initial strategies were dependent on the specific cases and unsophisticated. Most students focused on drawing all 9 stair steps and then counting. Some students said that in order to find the total number of blocks needed to build a staircase of 9 steps, you just found the sum of $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$. If you just add the numbers in their original order, then you will get the total. Many groups recognized that to find the total number of blocks needed in a staircase, you just add on the new length to the previous staircase total. Of course, this requires you to know the previous sum, so the students did not have an efficient way to get past this obstacle. One of the two invented algorithms is the pairing strategy. Frank and Maria looked for ways to add the numbers other than in the original order, and soon discovered the idea of pairing numbers to obtain common sums.

$$
\begin{aligned}
&1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 \\
&= (1 + 9) + (2 + 8) + (3 + 7) + (4 + 6) + 5 \\
&= 10 + 10 + 10 + 10 + 5 \\
&= 10 \times 4 + 5 \\
&= 45
\end{aligned}
$$

The symmetry of this approach appealed to many groups but they did not accept it as a superior solution strategy until the problem was extended to a larger number of stairs (Cai et al., in press).

Figure 2. The Staircase Problem

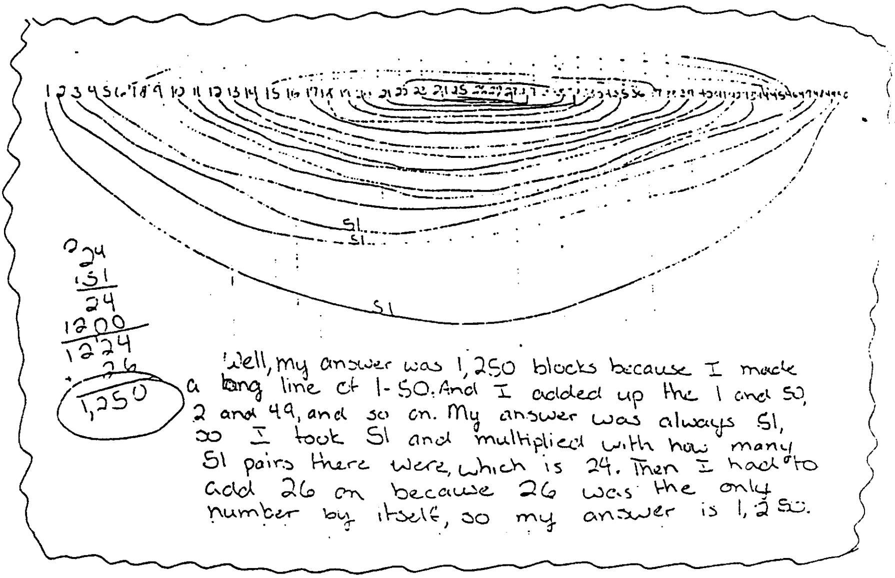
How many blocks are needed to build a staircase of 9 steps following the pattern in the figure below.



Show how you found your answer.
Try to find you answer in as many different ways as you can.

After all pairs had presented their solutions, the teacher asked the students: "What if we need to build a staircase of 50 steps, how many blocks do we need ?" After a pause, the teacher continued: "We have already discussed several ways to figure out the number of blocks needed to build a staircase of 9 steps. What if we used these ways to find out the number of blocks we need to build a staircase of 50 steps ? Show your solution to this problem in at least two different ways." The earlier algorithms of just counting by ones or adding in order were discarded by many students as too inefficient. Instead, this pairing algorithm became very popular. Jane in particular was very proud to share her work in finding the number of blocks needed to build a staircase of 50 steps with the class. Her work is shown in Figure 3.

Figure 3. Jane's Pairing Strategy



After Jane's presentation, many students were puzzled because Jane's answer was different from their answer. Some other students were puzzled because many of Jane's pairs were not equal to 51. Jane paired 25 with 27, 24 with 28, and so forth, but the sum of these pairs are 52, not 51 as Jane said. Cognitive conflict led to many constructive discussions in class. The teacher asked the students to investigate what happened in Jane's solution. Bill raised his hand and pointed out that when she did the pairing, Jane skipped 6. Then, Bill offered his correct pairing algorithm: "There are 25 pairs of 51. $51 \times 25 = 1275$. So the correct answer should be 1275." Some students also agreed with Bill that 1275 should be the correct answer.

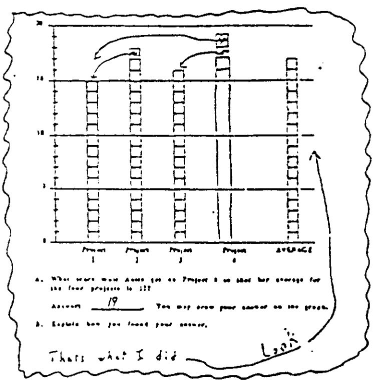
When Bill correctly paired the numbers, he did not have the unpaired middle number. That seemed to puzzle Jane. Jane asked, "Where is the middle one which is not paired with anybody?" Most of the students and the teacher in the class seemed to have difficulty understanding what Jane meant. The teacher asked, "What do you mean the middle number, Jane ?" Jane said: "Well, if you look at 9 steps, you paired the numbers, but the middle number 5 is not paired with anybody." Now the teacher understood what Jane meant. Since in 9 steps, there is a middle number 5 which is not paired, Jane appeared to believe that you should also have an unpaired middle number in 50 steps. So the teacher focused the class discussion on the resolution of Jane's question. The discussion ended at the point when students realized that if you have an even number of steps, you will not have an unpaired middle number. However, if you have an odd number of steps, you will have an unpaired middle number.

It should be indicated that the strategies that students invent, in many cases, are different from those usually taught in school. Strategies usually taught in school might force children to give up their own mathematical thinking and hinder their development of mathematical intuition (Kamii, 1994). For example, usually students are taught to use average formula either arithmetically or algebraically to solve the Sample Problem 7 shown above. Since $4 \times 17 = 68$. $68 - 15 - 18 - 16 = 19$. To use the average formula algebraically, let x = score Anita must get for her Project 4. $15 + 18 + 16 + x = 4 \times 17$. So the answer is 19 by solving the equation for x.

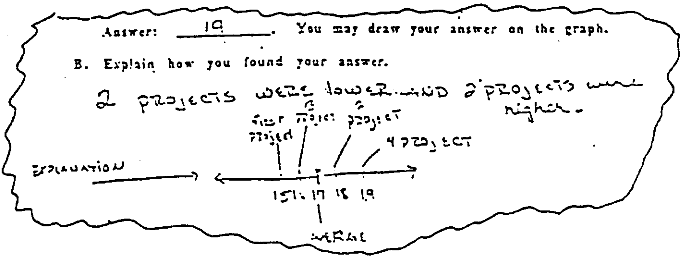
However, not all students would use these traditionally taught strategies. In fact, some students used "Leveling Strategy" and "Balancing Strategy" to solve the problem. Figure 4 shows examples of leveling and balancing strategies. These strategies are certainly different from those normally taught in school They suggest students' understanding of the averaging process.

Figure 4. <u>Sample Student Responses of Leveling and Balancing Strategies</u>

Leveling Strategy



Balancing Strategy



Final Remarks

In this article, it was argued that in order for students to have a conceptual understanding of mathematics, teachers should provide appropriate situations for them to learn through their own inventing. To nurture inventing, instruction should be designed according to students' thinking. Research shows that such instruction has a significant impact on students' learning of mathematics (Cai et al., 1997; Carpenter et al., 1989). This does not mean, however, that students can be left to invent everything for themselves. Rather, teachers should carefully choose appropriate situations and learning environments for students' knowledge construction process. Appropriate situations and learning environments can facilitate inventing and sense-making in classroom.

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