Three possible solutions for a problem of quadratic equation

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1. Introduction

In this paper, we will present a typical area problem of quadratic equation which appears in public examination[3] or textbook[4] very often. We found that the problem itself has a very high educational value because it could be difficult if students go in an inappropriate direction. The details of the problem statement as well as three suggested solutions will be presented in the next section.

2. Problem Statement

Refer to the diagram shown (see Figure 1) below. *ABCD* is a square with side 20 cm long. Given that $CM = CK = x$ cm and the area of $\triangle MAK$ is 102 cm^2, find the value of x.

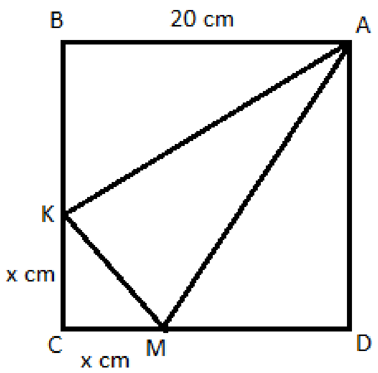

Figure 1

3 For instance, Question 11 of Hong Kong Examinations Authority (1998). *1998 HKCEE Mathematics Paper 2*. Hong Kong: Hong Kong Examinations Authority.

4 For instance, Question 21 & 22 of Exercise 3E in Leung, K. S., Wong, C. S., Hung, F. Y., Wan, Y. H., Wong, T. W., Ding, W. L. & Shum, S.W. (2014). *New Century Mathematics (2nd Edition) Book 4A*, p. 3.37. Hong Kong: Oxford University Press.

3. Suggested Solutions

The possible solutions to solve the concerned problem are shown as follows.

• Method 1

This is the easiest method and also indirect in nature.

Area of $\triangle MAK$

= area of square – area of $\triangle ABK$ – area of $\triangle AMD$ –area of $\triangle CKM$

$$102 = 20^2 - \frac{1}{2}(20-x)(20)\times 2 - \frac{1}{2}x^2$$

$$20x - \frac{1}{2}x^2 = 102$$

$x^2 - 40x + 204 = 0$

$(x-6)(x-34) = 0$

$x = 6$ or $x = 34$ (rejected as $0 < x < 20$)

• Method 2

For teaching purpose, we highly recommend students to try method 1 first. However, the following method can also be considered though it is complicated and involves much more tedious computation. The details are depicted as follows:

Draw AC cutting MK at E as shown (see Figure 2).

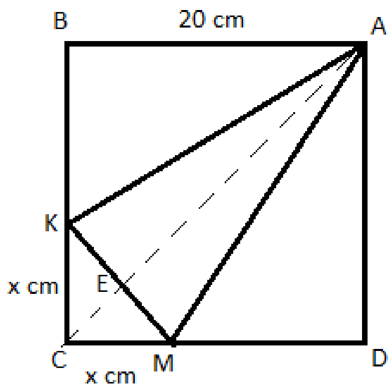

Figure 2

Now, $AB = AD = 20$ cm (property of square).

$BK = DM = (20 - x)$ cm

$\angle ABK = \angle ADM = 90^\circ$ (property of square)

$\therefore \triangle ABK \cong \triangle ADM$ (SAS)

$\therefore AK = AM$ (corr. sides, $\cong\triangle$s)

Similarly, we notice that:

$CE = CE$ (common)

$\angle KCE = \angle MCE = 45^\circ$ (property of square)

$CK = CM = x$ cm (given)

$\therefore \triangle KCE \cong \triangle MCE$ (SAS)

$\therefore KE = ME$ (corr. sides, $\cong\triangle$s)

Thus, $AE \perp KM$ (property of isos. $\triangle$)

$MK^2 = x^2 + x^2$ (Pyth. Theorem)

$MK^2 = \sqrt{2}x$ cm

By considering the area of $\triangle CKM$, we have

$$\frac{1}{2}x^2 = \frac{1}{2}(CE)(\sqrt{2}x) \Rightarrow CE = \frac{1}{\sqrt{2}}x \text{ cm.} \quad (\because x \neq 0)$$

Next, $AC^2 = 20^2 + 20^2$ (Pyth. Theorem)

$AC = 20\sqrt{2}$ cm

As $AE = AC - CE = \left(20\sqrt{2} - \frac{1}{\sqrt{2}}x\right)$ cm, we have

$$\frac{1}{2}(\sqrt{2}x)\left(20\sqrt{2} - \frac{1}{\sqrt{2}}x\right) = 102$$

$$20x - \frac{x^2}{2} = 102$$

$$x^2 - 40x + 204 = 0$$

$$x = 6 \quad \text{or} \quad x = 34 \text{ (rejected)}$$

- Method 3

There is another way to find AE by considering $\triangle AKE$. But then it will end up with an even more complicated solution. In order to alleviate the workload of students, we propose to rephrase the problem by giving more guidance to the students. The new problem statement and its solution are depicted as follows:

* Problem: Let $f(x) = x^4 - 80x^3 + 1600x^2 - 41616$

(a) (i) Show that both $(x - 6)$ and $(x - 34)$ are the factors of $f(x)$.

Solution

$f(6) = 6^4 - 80(6)^3 + 1600(6)^2 - 41616 = 0$

By Factor Theorem, $(x - 6)$ is a factor of $f(x)$.

Similarly, we also note that

$f(34) = 34^4 - 80(34)^3 + 1600(34)^2 - 41616 = 0$

and so $(x - 34)$ is a factor of $f(x)$.

(ii) Hence, factorize $f(x)$ as a product of 2 linear factors and 1 quadratic factor.

Solution

By (a), we notice that $(x - 6)(x - 34) = x^2 - 40x + 204$ is a factor of $f(x)$.

By long division, we get

$f(x) = (x - 6)(x - 34)(x^2 - 40x - 204)$.

(b) (i) Express AE, DE and AE in terms of x respectively.

Solution

$AK^2 = 20^2 + (20 - x)^2$ (Pyth. Theorem)

$AK = \sqrt{800 - 40x + x^2}$.

$MK = \sqrt{2}x$ cm (proved in method 2)

So, $KE = \dfrac{MK}{2} = \dfrac{\sqrt{2}}{2}x$ cm.

$AE^2 + KE^2 = AK^2$ (Pyth. Theorem)

So, $AE^2 = 800 - 40x + x^2 - \dfrac{1}{2}x^2 = 800 - 40x + \dfrac{x^2}{2}$

i.e., $AE = \sqrt{800 - 40x + \dfrac{x^2}{2}}$ cm.

(ii) Using the result obtained in (a), find the value of x.

Solution

$$\frac{1}{2}(AE)(MK) = 102$$

$$\Rightarrow \quad \frac{1}{2}\left(\sqrt{800-40x+\frac{x^2}{2}}\right)\left(\sqrt{2}x\right) = 102$$

$$\Rightarrow \quad \sqrt{2x^2\left(\frac{x^2}{2}-40x+800\right)} = 204$$

$$\Rightarrow \quad \sqrt{x^4-80x^3+1600x^2} = 204$$

i.e. $x^4-80x^3+1600x^2-41616 = 0$

By (a), we have $f(x) = 0$

$\Rightarrow$ $x - 6 = 0$ or $x - 34 = 0$ or $x^2 - 40x - 204 = 0$

$\therefore$ $x = 6$ or $x = 34$ (rejected) or $x \approx 44.6$ (rejected) or $x \approx -4.58$ (rejected)

The result follows.

4. Concluding Remarks

It is not a good pedagogical practice to force our students to cram for rigid solutions. Instead, we should provoke them to think more and try out different strategies to solve the same problem. Among the three methods presented, method 1 is the best as it is short and simple. Method 2 is good for helping students revise materials about congruent triangles. It is noteworthy that we can also help our students to figure out that by considering the symmetry of the diagram. In fact, students can also find by just regarding it as half of the diagonal of a square having side x cm. Method 3 is the most complicated after reframing the problem, but it also helps us to assess more skills for our students as Factor Theorem from the topic of polynomials is involved.

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