Reflections on teaching sine and cosine formulae in solving triangles

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Introduction

In the topic "More about trigonometry", most mathematics textbooks in Hong Kong will present in this order: the use of the sine formula in the A.S.A. (or A.A.S.) triangle, the use of the sine formula in the S.S.A. triangle (the ambiguous case), the use of the cosine formula in the S.A.S. triangle, and then the use of the cosine formula in the S.S.S. triangle. After that, a lot of drillings and practices will be put in the exercises on solving triangles (i.e. finding the lengths of 3 sides and the sizes of 3 internal angles of a triangle) and applications to 2-dimensional problems.

However, is it the best approach to introduce the topic? In this article, the writer will share some of the thoughts in introducing the concepts of these two formulae besides using the "textbook approach".

The sine formula

The formula for the area of a S.A.S. triangle $\frac{1}{2}ab\sin C$ can be introduced before the introduction of sine formula. The proof of this formula requires only junior form trigonometry, and by dividing *abc* from this formula, the sine formula can be proved.

For the A.A.S. triangle, it is enough to emphasize the use of sine formula to find the length of a side of triangle, and that it has one solution only.

For the S.S.A. triangle, most textbooks try to compare the length of a side of the triangle (without loss of generality, the length of "a" opposite to the angle "A") with the height of the triangle (the perpendicular distance from the vertex C to its opposite side) under different cases of the size of A, i.e. acute, right or obtuse. This approach is quite difficult for most average students because it creates a lot of noise to them. So, the writer will just emphasize that the sine formula is applied to find the size of the angle. (And later on, the writer will discuss the fact that the cosine formula can be applied as an alternative method to find the length of the third side.) For using the sine formula, there will be no solution when sin θ > 1 or angle sum of triangle > 180°. On the other hand, there will be 1 solution when only one angle with angle sum of triangle < 180°.

If there is time, the writer will discuss with students the condition for the "only one solution" for the S.S.A. triangle.

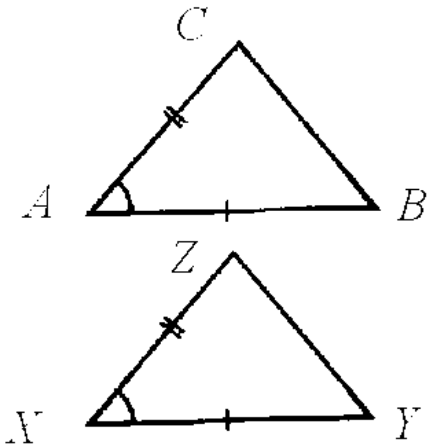

Figure 1

In the above figure, let $\angle BAC$ (or $\angle YXZ$) be the included angle ("A").

S_2S_1A is not a condition for congruence (that is, has more than 1 solution), where S_2, the side opposite to "A", is shorter than S_1, one of the "arms" of the included angle, i.e. $A < 90°$.

Counter example: AC = 6 cm, $\angle BAC$ = 30° and BC = 4 cm. Then, by sine formula, sin $\angle ABC$ = 0.75 and there will be two solutions ($\angle ABC \approx$ 48.5904° or 131.4096°).

The cosine formula

For the S.A.S. triangle, the cosine formula is used to find out the third side first. Then we can find the second angle by either sine formula or the cosine formula. The writer thinks that applying cosine formula twice is too complicated for students (especially when they have no programme input in their scientific calculator). Therefore, the writer will emphasize that even if sine formula is used, for the second angle and the third angle, we always find the smaller one to avoid ambiguity. That is, for a S.A.S. triangle, find out the third side first. Then,

- if the largest angle is given, find out the smallest angle.
- if the medium angle is given, find out the smallest angle.
- if the smallest angle is given, find out the medium angle.

Of course, after finishing this section, the writer will discuss with students which formula is more convenient and let them choose their own way. (The ambiguity verses the unique solution? The one without the need for a calculator programme verses the one requires two executions of one calculator programme - the programme for finding the side and the programme for finding the angle?)

For the S.S.S. triangle, most textbooks will emphasize the finding of the largest angle first by cosine formula (that is, the angle opposite to the longest side) to avoid ambiguity (because then the second angle and the third angle will not have two solutions even when sine formula is used). But then, they just ask the students to find the second angle by cosine formula again, and then the third angle by angle sum of triangle. Again, the writer thinks that applying cosine formula twice is too complicated for students (especially when they have no programme input in their scientific calculator). Therefore, again the writer will emphasize that even if sine formula is used, for the second angle and the third angle, we always find the smaller one to avoid ambiguity. That is, for a S.S.S. triangle, always find out the largest angle first by cosine formula. (For cosine formula, there will be no solution when cosine value < -1 or cosine value > 1.) Then, find out the smallest angle by sine formula. Finally, use angle sum of triangle to find the remaining angle (the medium angle).

The writer will also introduce the use of cosine formula for the S.S.A. triangle. A quadratic equation in the length of the third side will be formed and the number of roots will vary according to the discriminant of the quadratic equation formed. The aim of introducing this method is that when a question asks for the exact value of the third length, the cosine formula will give a root in surd form by using the quadratic equation formula. For abler students, the writer will also discuss the number of roots, i.e. for cosine formula, no solution when the discriminant < 0 or the two roots are both negative.

For cosine formula, 1 solution when product of roots < 0 (i.e. one of the roots is positive and the other root is negative).

For abler students, the writer will ask them if there is a need for creating a “tangent formula”. There is no urgent need for setting up such a formula because the existing sine formula and cosine formula are already enough for solving different types of (congruent) triangles we learnt in junior form. (Actually, there is such a “tangent formula for S.A.S. or A.S.A. or A.A.S. triangles. The law is $\dfrac{a-b}{a+b}=\dfrac{\tan\dfrac{A-B}{2}}{\tan\dfrac{A-B}{2}}$.)

Here is a summary of what I intended to discuss in the class.

Given	**3 sides (S.S.S.)**	**2 sides and 1 angle**		**1 side and 2 angles (A.S.A. or A.A.S.)**
		included angle (S.A.S.)	**opposite angle (S.S.A.)**	
Number of solutions	**0 or 1** (No solution when $\cos\theta < -1$ or $\cos\theta > 1$)	**1**	**0 or 1 or 2** (For sine formula, no solution when $\sin\theta > 1$ or angle sum of triangle $> 180°$. For cosine formula, no solution when discriminant < 0 or the two roots are negative.) 1 solution when only one angle with angle sum $< 180°$ or product of roots < 0.	**1**
Method of solutions	**cosine formula** to find the largest *angle*, then use sine formula (easier but need to check) or cosine formula (better but a bit complicated) for the S.S.S.A. triangle	**cosine formula** to find the *length*	**sine formula** (if exact value of the third side are NOT required) to find the *angle* or **cosine formula** (if exact value of the third side are required) to find the *length*	**sine formula** to find the *length*

Figure 2

I would like to end by discussing the common errors in two questions appeared in the final mathematics exam paper in my school.

Question 1: In the figure, *PRS* is a straight line, *PQ* = 10 cm, *QR* = *QS* = 7 cm and $\angle QPR = 35°$. Find the size of $\angle PQR$.

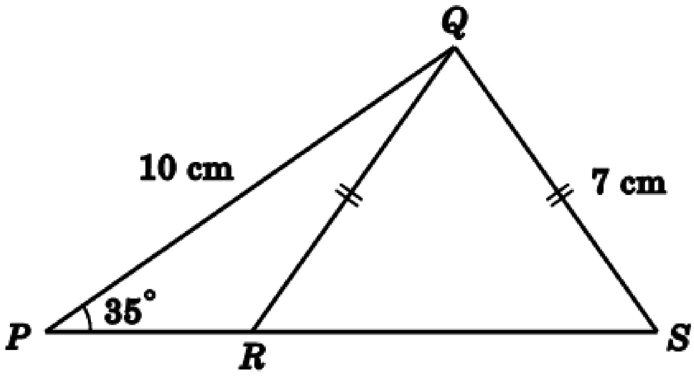

Figure 3

Here is the marking scheme.

In $\triangle PQR$, by the sine formula, $\dfrac{7\text{ cm}}{\sin 35°} = \dfrac{10\text{ cm}}{\sin \angle PRQ}$	1M
$\angle PRQ = 55.0243°$, *corr. to 6 sig. fig.*	
or $\angle PRQ = 180° - 55.0243° = 124.976°$	1A
$\because$ Given *PSR* is a st. line and ΔQRS is isos., so $\angle PRQ$ is an obtuse angle.	
$\therefore$ $\angle PRQ = 124.976°$	
$\angle PQR = 180° - 35° - 124.976°$ $\angle$ sum of $\triangle$	
$= \boxed{20.0243} = \underline{\underline{20.0°}}$, *corr. to 3 sig. fig.* (NOT accept 89.9757°)	1A

A lot of students are having difficulty in finding the size of $\angle PRQ$. They just give 55.0243° as their answer, without considering the angle in the second quadrant.

Question 2: In the figure, ship *A* and ship *B* are 15 km apart. The bearings of ship *A* and ship *B* from the lighthouse *L* are S50°E and N60°E respectively. The bearing of ship *B* from ship *A* is N25°W.

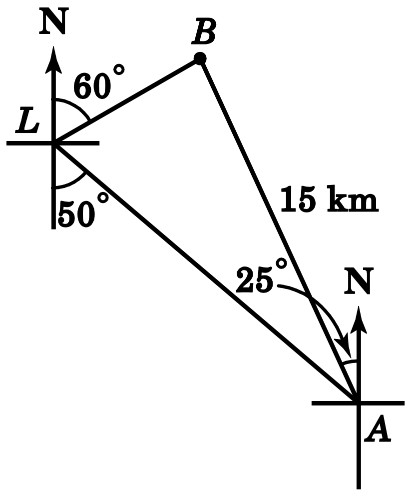

Figure 4

(a) Find the distance between ship B and lighthouse L.

(b) John claims that the distance between ship A and lighthouse L is greater than that between ship B and lighthouse L. Do you agree with John? Explain your answer.

Here is the "initial" marking scheme.

(a)	$60° + \angle BLA + 50° = 180°$	
	$\angle BLA = 70°$	1A
	$25° + \angle BAL = 50°$	
	$\angle BAL = 25°$	or 1A (either one)
	In $\triangle BAL$, by the sine formula, $\dfrac{BL}{\sin\angle BAL} = \dfrac{AB}{\sin\angle BLA}$	1M
	$BL = \dfrac{15\sin 25°}{\sin 70°}$ km = 6.746114406 km	
	$\therefore$ The distance between ship B and lighthouse L is 6.75 km.	1A
(b)	In $\triangle BAL$, $\angle LBA + 25° + 70° = 180°$	
	$\angle LBA = 85°$	1A
	By the sine formula, $\dfrac{AL}{\sin\angle LBA} = \dfrac{AB}{\sin\angle BLA}$	1M
	$AL = \dfrac{15\sin 85°}{\sin 70°}$ km = 15.9019 km, *corr. to 6 sig. fig.*	
	which is greater than BL.	
	$\therefore$ John's claim is agreed.	1A must with explanation

I say it is an "initial" marking scheme because after marking nearly 190 scripts of exam papers, only one girl, who is not good in the past mathematics examinations, can explain by using the fact that "larger angle, larger side". Here is what she writes.

$\angle LBA = 180° – 25° – 70° = 85° > 25° = \angle BAL$, so $AL > BL$.
And hence John's claim is agreed.

Note that the longest side is opposite to the largest angle, and vice-versa. Therefore, when answering part (b), actually there is no need to calculate the length of *AL* by sine formula. I have included her solution in the finalized marking scheme, so that other students can think if there can be more than one method of solution to the same problem. Sine formula and cosine formula sometimes may not be the most powerful method!

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