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A Case Study on Teaching and Learning of Quadratic Factoring

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Introduction

Factoring quadratic trinomial, which requires to rewrite $ax^2 + bx + c$ into the form $(mx + n)(px + q)$, is one of the most difficult algebraic challenges for students of the high school mathematics curriculum. Kieran (2006) points out that based on the research on investigating of "visual salience"[1] in the learning of algebra, students perform significantly better in recognition tasks involving visual salient rules. Unfortunately, quadratic factoring such as $x^2 + 8x + 12 = (x + 2)(x + 6)$ is not visual salient. Therefore students fail to develop reasoning in non-visual salient algebraic expression and cause poor performance in quadratic factoring. Leong et al. (2010) have similar point of view. They state that students' perception of the cross method as arbitrary and the subsequent failure to make sense leading significant number of students cannot use cross method effectively, even after careful demonstration through repeated examples has been delivered to students. Since most of the teachers find that quadratic factoring is difficult to teach, and most of the students feel that quadratic factoring is difficult to understand at the same time, therefore many students attempt to learn algebra by using memorization, and many teachers use direct-instruction methods to encourage memorization too (Leitze and Kitt, 2000). But direct-instruction approach always leads to undesirable result. Didis et al. (2011) highlight that even students know some rules related to solving quadratics, they apply the rules without realizing why they did so, and do not think about their own works are

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1 Visual salient rules have visual coherence that makes the left- and right-hand sides of the equations appear naturally related to one another. For example, $(x^y)^z = x^{yz}$ is considered a visually salient rule while $x^2 - y^2 = (x + y)(x - y)$ is not (Kieran, 2006).

whether mathematically correct or not too. It means that students' understanding is procedural rather conceptual.

Quadratic Factoring by Cross Method

In Hong Kong, students mainly perform quadratic factoring by cross method. Although cross method has been used for a long time in the curriculum of mathematics education in Hong Kong, Chan (2004) points out that most of the less-able students fail to apply the cross method. Now I would like to perform a brief study to investigate why student fails to apply cross method to perform quadratic factoring in some situations. Peter was a fifteen years-old student in a secondary school studying in form four. His overall performance in mathematics was above average (ranked top 30 out of 180 students in last year). A pre-test, which was about factoring quadratic trinomials, was delivered to his class before starting a new chapter: quadratic equations. He factorized all the eight questions in the form $x^2 + bx + c$ successfully but failed to factorize all the seven questions in the form $ax^2 + bx + c$ (where $a \neq 1$). He had already learnt how to perform quadratic factoring by using cross method in last year and he seemed to forget this skill. A remedial class was held after the test and two questions had been given to Peter at the beginning. The first question was "factorize $x^2 + 5x + 4$" and the second one was "factorize $4x^2 + 5x + 1$". Here is the conversation [2]:

Episode 1: Quadratic factoring by cross method

[N1] Teacher: Peter, can you factorize $x^2 + 5x + 4$?

[N2] Peter: *(Firstly he writes down the solution in the form (x +)(x +) and keep calculating in his mind, finally he writes down the answer (x + 1)(x + 4) after ten second.)* … I have done it.

[N3] Teacher: Why do you know that the answer is $(x + 1)(x + 4)$?

[N4] Peter: Because one times four is equal to four at the end and one plus four is equal to five in the middle of the expression.

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2 This and all the subsequent segments of transcripts have been translated from Cantonese by the author.

[N5] Teacher: Great. Now can you factorize $4x^2 + 5x + 1$?
[N6] Peter: *(Hesitate about ten second)* … I don't know.
[N7] Teacher: Why?
[N8] Peter: I don't know how to handle $4x^2$.
[N9] Teacher: Why can't you handle the term $4x^2$?
[N10] Peter: I don't know how to factorize $4x^2$ …
[N11] Teacher: Never mind. Do you know the meaning of "$4x^2$" ?
[N12] Peter: It is 4 times x^2 .
[N13] Teacher: So you know how to factorize the number 4 but not x^2 ?
[N14] Peter: Yes.

At the beginning of the episode, Peter recalled the multiplication table from his long-term semantic memory[3], retrieved the corresponding entities finding the possible combination of two factors for $+ 4$. The routine manipulations were occurred and nothing else was needed. At the same time Peter also focused on the term $+ 5x$ and keep finding which combination gave a sum of $+ 5$ in his mind. But for the x^2 term, Peter might just be manipulating the symbols x^2 in a routine way without actually giving any thought to the meaning of it. His expectation was only focused on finding the solution in the form of $(x + a)(x + b)$ at the end of the process. He was able to find out that $x^2 + 5x + 4 =$
$(x + 1)(x + 4)$ with explanation why he chose to express the constant term "4" into "1×4" rather "2×2" . But later on Peter failed to perform the similar factoring, which was $4x^2 + 5x + 1$. Why did he fail apply the same strategy he used before to solve the problem this time? It was because he failed to perform algebraic manipulations with the term "x^2" . Hence he did not know how to factoring the numerical value "4" with the existence of "x^2" . It seems to be difficult for teachers to imagine why some students, just like Peter, cannot handle the term "$4x^2$" . French (2002) uses the term

3 Long-term semantic memory stores general knowledge not identified by a timeline for when the event occurred. In mathematics, semantic implies the ability to access knowledge over other knowledge based upon context (Kotsopoulos, 2007).

"juxtaposition"[4] to describe the algebraic symbols just like "$4x^2$", "$2p$" even "2^3" and he highlight that juxtaposition can be given a variety of different interpretations in different contexts. For example, "$4x^2$" can be understood as $4 \times x^2$, but it can also be understood as $4 \times x \times x$ or $x^2 + x^2 + x^2 + x^2$. Many teachers assume that students interpret "$4x^2$" as $4 \times x \times x$ when performing quadratic factoring by cross method. But it is not difficult to imagine that some of the students tend to interpret "$4x^2$" as $4 \times x^2$ – which means that they may interpret x^2 as a "non-separable" whole object. They know they are required to factorize 4 into 1×4 or 2×2, but they may never intend to factorize x^2 into $x \times x$.

But why Peter knew that he needed to factorize x^2 into $x \times x$ in the first question (i.e. $x^2 + 5x + 4$)? Sfard and Linchevski (1994) suggest that this situation was a typical example of pseudo-structural thinking: Peter was able to handle some kind of mathematical objects, but his thinking was completely inflexible and the appropriate kind of structural interpretation was unavailable. He knew that he needed to factorize x^2 into $x \times x$ in order to express the solution in the form of $(x + a)(x + b)$. But he might not know why it was required to do so conceptually. It means that he might not reify[5] the concept of quadratic factoring by himself; he might still be able to perform the process, but his understanding remained instrumental (procedural). Since the conceptual understanding of quadratic factoring had not been developed by Peter, therefore he could solve simple quadratic factoring $x^2 + 5x + 4$ procedurally but could not handle different situations such as factoring $4x^2 + 5x + 1$. Sfard and Linchevski (1994) state that teachers should fight against such kind of pseudo-structural conceptions developed by the students:

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4 Juxtaposition is the placement of two things (usually abstract concepts, though it can refer to physical objects) near each other.

5 Reification, a transition from an operational to a structural mode of thinking, is a basic phenomenon in the formation of a mathematical concept. In fact, reification brings a mathematical object into existence and thereby deepens our conceptual understanding (Sfard, 1994).

"The students may easily become addicted to the automatic symbolic manipulations. If not challenged, the pupil may soon reach the point of no return … it seems very important that we try to motivate our students to actively struggle for meaning at every stage of the learning.

Furthermore Sfard and Linchevski (1994) suggest that there are two more kinds of thinking in manipulating algebra: operational thinking – the lowest level which deals with arithmetical processes only; another is structural thinking – the highest level which deals with formal algebraic manipulations. Pseudo-structural thinking is the transition in-between them. Sfard and Linchevski point out that even the same representation, the same mathematical concepts, may sometimes be interpreted as processes and at other time as objects. When the concepts are interpreted as processes, the operational thinking is involved; when they are interpreted as objects, then the structural thinking is involved. For example, using concrete numbers instead of general coefficients and presenting the solution in the form of verbal prescription is purely operational rather than structural. Quadratic factoring ties the arithmetical processes (involve factoring of coefficients) and the formal algebraic manipulations (involve understanding of $ax^2 + bx + c$) together. Cross method mainly focus on seeking the correct combination of the factors of the x^2 term and constant term, it means that cross method involves operational thinking intensively. But cross method itself is not effective to facilitate a transition from an operational to a structural mode of students' thinking, which limits the reification of quadratic factoring by students. Fail in reification made students fail to develop conceptual understanding of quadratic factoring.

Quadratic Factoring by Grouping Method

To overcome the difficulty of learning quadratic factoring for some students, a different method rather than cross-method should be applied. Chan (2004) suggests to apply the grouping method [6], which has been widely used in

6 The grouping method will be called as the "A-C Method", "X-Box Method" even "Diamond Method" in the teaching materials of United States.

mathematics education in United States. For example, some teaching materials demonstrate the factoring of $2x^2 + 5x + 3$ (where $a = 2$, $b = 5$ and $c = 3$) by grouping method as follows:

Step	Procedure	Example
1.	Find the product ac	$(2)(3) = 6$
2.	Express ac as multiple of two factors	$6 = 1 \times 6$; $6 = 2 \times 3$
3.	Sum each pair of two factors	$1 + 6 = 7$; $2 + 3 = 5$
4.	Which pair of factors can give a sum equal to b? Rewrite b as the sum of these factors	$2x^2 + 5x + 3$ $= 2x^2 + 2x + 3x + 3$
5.	Group the expression into two parts and find the common factor of each group if any	$2x^2 + 2x + 3x + 3$ $= 2x(x + 1) + 3(x + 1)$
6.	Take out the common factor of two parts again and write the answers	$2x(x + 1) + 3(x + 1)$ $= (2x + 3)(x + 1)$

Obviously grouping method can be easy to be follow by students since it has a clear flow of procedures. Also there is an advantage of using grouping method compared with cross method: for grouping method, students can apply the prior knowledge of algebra they have learnt in lower form, which is factoring polynomials by using common factors and grouping of terms. Although grouping method mainly emphases on the procedures, it can demonstrate that factoring is a reverse process of expansion more directly and logically compare with cross method. After a few practices students may realize the relation between quadratic expansion and factoring by observing the intermediate steps in the working $2x^2 + 5x + 3 = 2x^2 + 2x + 3x + 3 = 2x(x + 1) + 3(x + 1) = (2x + 3)(x + 1)$, and they may develop the relational (conceptual) understanding and the logical understanding [7], with the procedural understanding throughout the process. Of course, for facilitating students'

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7 Logical understanding concerns the relationship of implication between the successive statements. It is a kind of understanding besides instrumental (procedural) understanding and relational (conceptual) understanding under the model of intelligence introduced by Skemp (1987).

better understanding of grouping method, teachers may prove why these procedures work (especially using the product ac to find the correct combination of two factors). According to the study of Chan (2004), grouping method is better than cross method, at least for less-able students.

Quadratic Factoring by Tool-based Geometric Approach

Besides the cross method and grouping method methods, Leitze and Kitt (2000) highlight that using concrete models as a tool-based pedagogy to introduce concepts, rather than concentrate only on the abstract or symbolic manipulations, can facilitate better understanding of students to algebra. Similarly Leong et al. (2010) suggest that algebra tiles rather than cross method should be introduced in teaching quadratic factoring, since algebra tiles could be perceived as sensible and non-arbitrary to the students. They have shown that algebra tiles can provide a more concrete and visual representation, and allow students to make geometric sense alongside the algebraic manipulation in quadratic factoring. Here is the demonstration of factoring $2x^2 + 5x + 3$ by using algebra tiles:

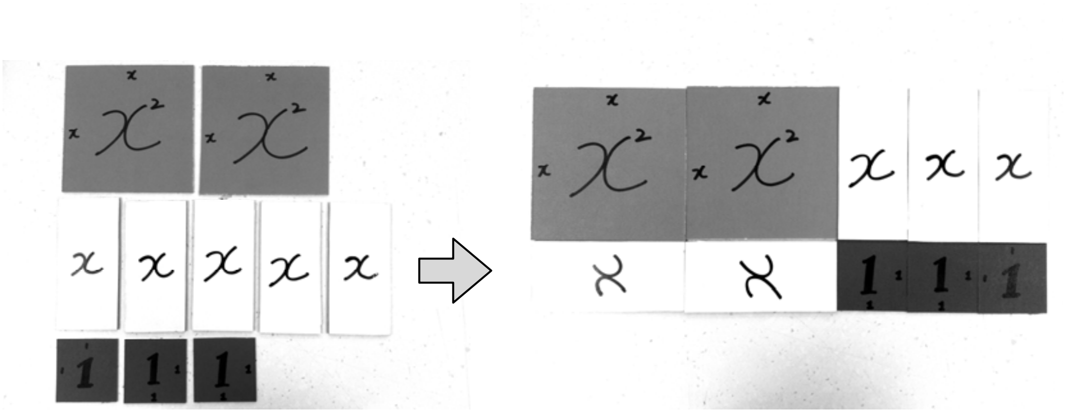

Figure 1: Factoring $2x^2 + 5x + 3$ as the product $(2x +3)(x + 1)$ geometrically

As there is still lack of studies about students using algebra tiles to learning quadratic factoring, therefore I would like to study how this tool-based geometric approach can work in practice. After Peter failed to perform quadratic factoring by cross method, the method of using algebra tiles was introduced to him. A set of worksheets [8], which illustrated the use of algebra tiles to perform quadratic factoring, had been used in the remedial class. Apart from the procedure of using algebra tiles, the concept underlying the procedure (forming rectangle as factoring quadratic trinomial) had also been introduced to Peter briefly. But in the worksheets, only factoring of quadratic trinomial with positive coefficients had been introduced (which meant that the values of a, b and c are all positive in the trinomial $ax^2 + bx + c$). After a few practices with the new method, Peter was asked to answer three questions : "factorize $2x^2 + 7x + 3$ ", "factorize $4x^2 + 12x + 9$ " and "factorize $6x^2 + 13x + 6$ ":

Episode 2: Quadratic factoring by using algebra tiles

[N15] Teacher: Now can you factorize $2x^2 + 7x + 3$ by the algebra tiles on the table?

[N16] Peter: I'll try … *(arrange the algebra tiles as the form of rectangle [refer to Figure 2] after 20 seconds)* … it is done.

[N17] Teacher: What is the answer?

[N18] Peter: It should be $(x + 3)(2x + 1)$.

[N19] Teacher: Why do you know that one of the factor is $(x + 3)$?

[N20] Peter: Because one of the length of the rectangle is $x + 3$.

[N21] Teacher: Why is it $x + 3$?

[N22] Peter: There are one "x^2" tiles and three "1" tiles on this side, so its length is $x + 1 + 1 + 1 = x + 3$.

[N23] Teacher: So you find that another width is $x + x + 1$, which is $2x + 1$, right?

[N24] Peter: Yes.

8 The worksheets were extract form Loh C.Y. (1984).

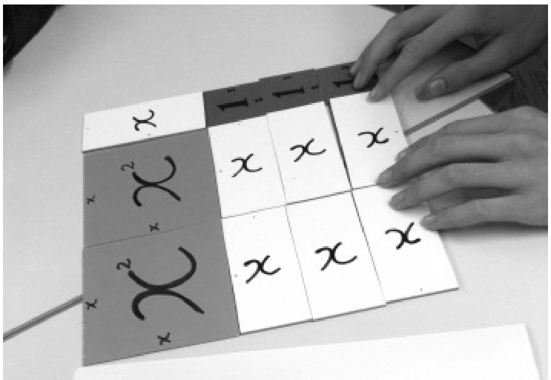

Figure 2 : Factoring $2x^2 + 7x + 3$

[N25] Teacher: Why we need to arrange the algebra tiles as rectangle in factoring?

[N26] Peter: Ummm … just like when we factorize the number 6 as 2×3 , we can think that we have a rectangle with area 6, then 2 and 3 are its length and width. It is factorization.

[N27] Teacher: So you think that we can perform similar technique in factoring quadratic trinomial?

[N28] Peter: … I think yes.

[N29] Teacher: Now can you factorize $4x^2 + 12x + 9$?

[N30] Peter: I'll try … *(arrange the algebra tiles as the form of rectangle [refer to Figure 3] after 30 seconds)* … it is done.

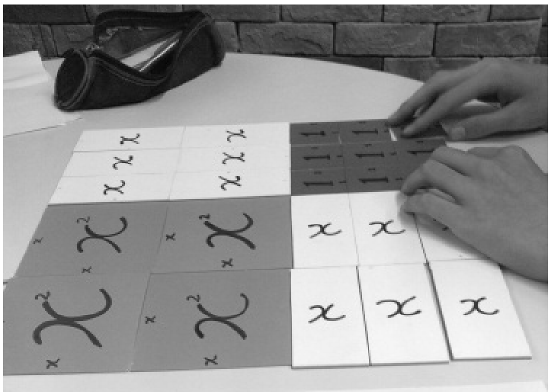

Figure 3: Factoring $4x^2 + 12x + 9$

[N31] Teacher: What is the answer?

[N32] Peter: It should be $(2x + 3)(2x + 3)$.

[N33] Teacher: Great. Can you draw a rough sketch of algebra tiles on the table now?

[N34] Peter: Do I require drawing all the tiles out?

[N35] Teacher: Up to you.

[N36] Peter: *(30 seconds later, Peter drew the rough sketch of the algebra tiles)* It is done.

[N37] Teacher: Well done. Now can you factorize $6x^2 + 13x + 6$ by a rough sketch of algebra tiles only? This means that you cannot actually touch the algebra tiles this time.

[N38] Peter: *(30 seconds later, Peter completes the sketch [refer to Figure 4])* … It is done.

[N39] Teacher: So what is the answer?

[N40] Peter: $(2x + 3)(3x + 2)$.

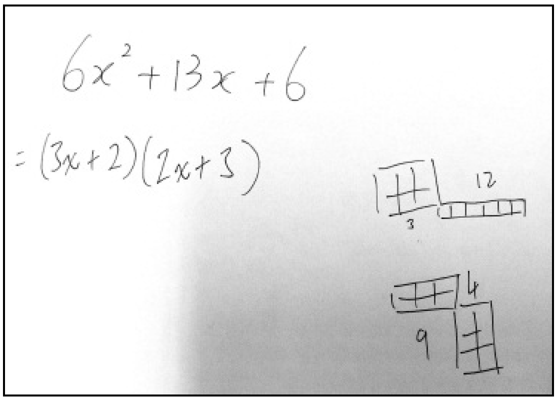

Figure 4: Factoring $6x^2 + 13x + 6$

In this episode, algebra tiles were acted as visual mediators discovering the mental process of Peter. It showed that Peter understood the meaning of quadratic factoring under the geometric representation of the algebra tiles: he knew that the action of arranging algebra tiles into rectangles was actually factoring the quadratic trinomials. It is believed that he probably reified the geometric representation – he understood that the operation of forming rectangle from algebra tiles was structurally identical to quadratic factoring. Recall that reification is a warranty of conceptual understanding (Sfard and Linchevski, 1994), therefore it can explain why Peter could handle quadratic factoring in different situations. Furthermore, Sfard and Linchevski (1994) highlight that reification which consists of the rises in the degree of abstraction and generality, will result in facilitating the performance rather than adding complexity. Although reification itself may be difficult to achieve, once it happens, its benefits become immediately obvious – the decrease in difficulty and the increase in manipulability is immense. Back to our case, Peter needed to use the algebra tiles to perform quadratic factoring and he felt a little bit difficult to manipulate with the algebra tiles initially. But later on (after a few exercises) he had built up a mental visual representation of the algebra tiles and performed the factoring with the rough sketch only. At the same time, the duration for Peter to solve a quadratic factoring problem was shorten significantly.

How about Negative Coefficients?

Every method has its own strengths and weaknesses. For the tool-based geometric approach, the weakest link is dealing with negative coefficients. Verschaffel, et al. (2006) state that there is a continuing debate whether negative numbers should be introduced through concrete representation or as formal abstraction. Henderson (1994) also highlights that the Greeks needed to rewrite the quadratic equations to avoid the presence of negative coefficients before solve the equations geometrically in the past. One of the reasons maybe that negative quantities are difficult to be understood when they are represented as geometric representation. Furthermore Leong et al. (2010) show that using algebraic tiles did not lead itself intuitively to “negative areas”, and the transition involving a gradual downplaying of the geometric significance (the idea of area) and the increasing emphasis on algebraic manipulation (checking the products and simplification of like *x*-terms) should be introduced. After introducing the idea of negative area, they find that a significant number of students have successfully factorized quadratic trinomials involving negative coefficients such as
$x^2 - 2x - 3$. But in my case study, I did not introduce the concept of “negative area” to Peter directly. Instead I had tried to investigate whether Peter could develop the concept of “negative area” by himself or not. Three questions had been given to Peter: “factorize $10x^2 + x - 2$ ”, “factorize $4x^2 - 11x - 3$ ” and “factorize $8x^2 + 18x - 5$ ”.

Episode 3: Using algebra tiles to factorize quadratic trinomials with negative coefficients

[N41] Teacher: Peter, now can you factorize $10x^2 + x - 2$?

[N42] Peter: *(After trying about 10 seconds)* … it seems impossible.

[N43] Teacher: Why is it impossible?

[N44] Peter: It is impossible to form a rectangle with the given algebra tiles.

[N45] Teacher: Yes. It seems that it is impossible to form a rectangle with only one "x" tiles this time. Now I would like to give you some hints: you are required to use extra "x" tiles. You can take any number of "x" tiles you like.

[N46] Peter: Any number of extra "x" tiles?

[N47] Teacher: Yes. But I think about eight to ten "x" tiles are enough for you to find the answer out.

[N48] Peter: You mean that I need to do subtraction this time?

[N49] Teacher: You may say that … but I advise you to focus the "x^2" tiles and "1" tiles first, their numbers are fixed.

[N50] Peter: Alright … *(after 30 seconds)* … I find the answer. It should be $(5x - 2)(2x + 1)$ *[refer to figure 5]*.

Figure 5: Factoring $10x^2 + x - 2$

[N51] Teacher: Why do you know that $10x^2$ should be arranged as $5x \times 2x$ rather than $10x \times 1x$?

[N52] Peter: If I arrange 10 "x^2" tiles into a row of ten tiles, the difference will be too large.

[N53] Teacher: What difference?

[N54] Peter: The numbers of "x" tiles on the top-right and bottom-left corners.

[N55] Teacher: Why do you know that we should subtract "x" tiles in this question?

[N56] Peter: Because of the "–2" in $10x^2 + x - 2$.

[N57] Teacher: So which tiles represent "–2" on the desk?

[N58] Peter: *(Point to the two "1" tiles on the bottom-right corner)* These two.

[N59] Teacher: But both of these two "1" tiles have the area of 1, not -1 . You mean that the area of these tiles should be -1 this time?

[N60] Peter: *(Hesitate about 10 seconds)* … I cannot sure …

[N61] Teacher: That's fine, your answer is correct. Shall we start a new question?

[N62] Peter: Okay.

[N63] Teacher: Can you factorize $4x^2 - 11x - 3$? Similar to the pervious question you can use any number of "x" tiles.

[N64] Peter: I'll try it out … *(after 20 second)* … It is done. The answer should be $(4x + 1)(x - 3)$ *[refer to figure 6]*.

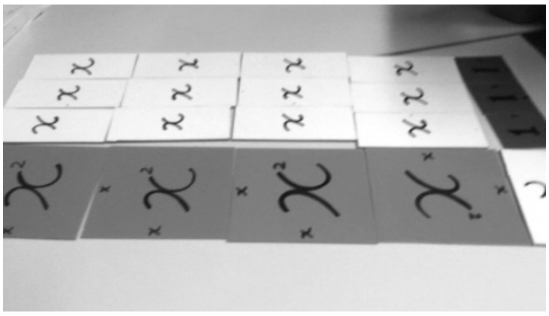

Figure 6: Factoring $4x^2 - 11x - 3$

[N65] Teacher: Why is one of the factors $x - 3$?

[N66] Peter: Because there are one "x^2" tiles and three "1" tiles on this side.

[N67] Teacher: But it should be $x + 3$ rather than $x - 3$, unless … do you mean that the length of this side of "1" tile is -1 ?

[N68] Peter: … It seems yes.

[N69] Teacher: But is it possible for a square to have a negative length?

[N70] Peter: *(Hesitate about 10 seconds)* … I cannot sure …

[N71] Teacher: Have you ever seen a figure with negative lengths?

[N72] Peter: No.

[N73] Teacher: That's fine. Anyway your answer $(4x + 1)(x - 3)$ is correct again. Let's go for the last question. Can you factorize $8x^2 + 18x - 5$ by a rough sketch of algebra tiles only this time?

[N74] Peter: *(30 seconds later, refer to figure 7 for the sketch)* It is done.

[N75] Teacher: So what is the answer?

[N76] Peter: $(4x - 1)(2x + 5)$.

[N77] Teacher: It is correct. Thank you Peter, you have done a great job.

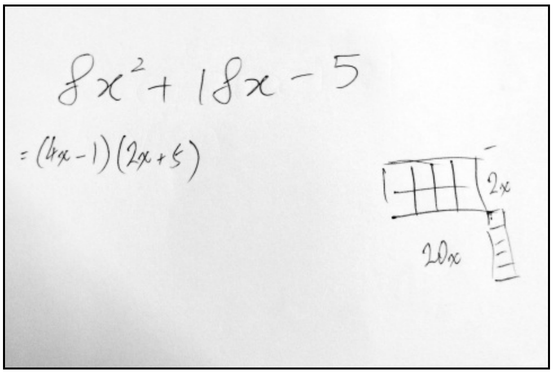

Figure 7: Factoring $8x^2 + 18x - 5$

Throughout this episode, the concept of negative area had never been introduced by the teacher. In factoring $10x^2 + x - 2$, Peter realized that he needed to perform subtraction of "x" tiles after extra tiles had been given to him. At this point he treated the "x" tiles as concrete object, and discovered that $5x - 4x = x$ from the geometric representation. By the way he gave $10x^2 + x - 2 = (5x - 2)(2x + 1)$ but he could not explain why the two "1" tiles could represent "–2" instead of "2" . It might show that Peter had neither realized the existence of negative area nor negative length in the geometric representation. He even did not realize that he had already performed a factoring of the negative constant: $-2 = 1 \times (-2)$. The working was suddenly come up from his mind and it is difficult to identify whether his understanding was procedural or conceptual. When Peter tried to factorize $4x^2 - 11x - 3$ later, his geometric representation came up with negative length first time. Once the teacher asked Peter the meaning of the negative length, he seemed to be frustrated and could not give clear explanation. In the last

question, Peter only drew the $8x^2$ and -5 in form of algebra tiles, wrote the two terms "$20x$" and "$2x$", marked the minus sign at the top of the term "$2x$", then he realized that $20x - 2x = 18x$ and came up with the solution $8x^2 + 18x - 5 =$
$(4x - 1)(2x + 5)$. This time, Peter only drew half of the structure of geometric representation and completed another half by algebraic representation. It showed that Peter proceeded a jump from operational to structural thinking and he had reified the geometric representation of quadratic factoring again – although he still did not make clear about the concept of negative length. However it is inappropriate to classify Peter's understanding of geometric representation with negative lengths as pseudo-structural conception. The reason is that a re-test has been delivered to Peter approximately one month later after the remedial class. The format and the time limit remain the same from the previous pre-test. This time Peter has answered all the fifteen questions correctly by using geometric representation to perform quadratic factoring.

Further Use of Algebra Tiles

Algebra tiles are useful in our scenario, but we still need to consider a way to facilitate smooth transition from positive to negative coefficients in practice if algebra tiles are used in the classroom. A possible way is the modification of the "x" tiles: The "x" tiles in white originally can be flipped over to become "$-x$" tiles in black, just like Othello (refer to figure 8). Once the students have raised the concept of negative quantities in geometric representation, what teachers need to do is just flipping the "x" tiles, introduced the concept of "$-x$" tiles briefly and then left the students to explore and construct their own understandings – they may be procedural and conceptual.

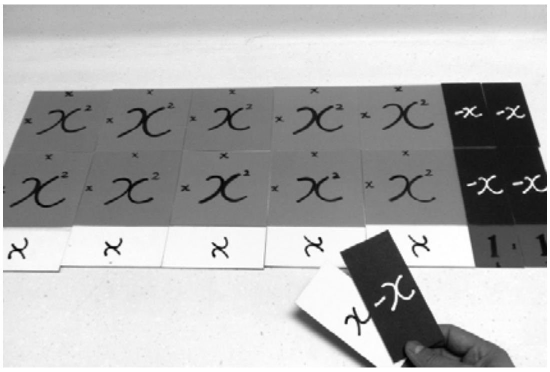

Figure 8: Using "x" and "$-x$" tiles to represent $10x^2 + x - 2$

If the weakest link of the algebra tiles can be overcome, it is possible to wider its uses on other topics related to quadratic expressions. Allaire and Bradley (2001) have already demonstrated the possibility to perform quadratic factoring, solving quadratic equations, formation of quadratic formula and completing the square by using algebra tiles. Furthermore it is possible to design group activities involving algebra tiles in order to maximize the benefit of collaborative learning. Algebra tiles can be acted as visual mediators in quadratic factoring, it means that students in the group can easily identify the objects (both coefficients and factors) and coordinate their communication by the algebra tiles. In collaborative learning, students can construct their own understandings through the discourse (Sfard 2001), and the role of teacher is to facilitate and guide them to the goal – develop conceptual understanding.

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