Classroom Capsule: Sphere, Cylinder and Bicone

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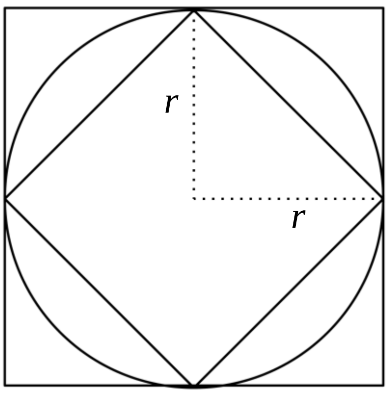
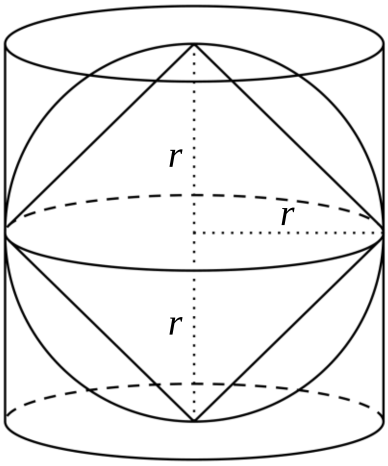
It is well known that circumscribed and inscribed regular polygons could be used to approximate the value of π. In the simple case of bounding a circle by its circumscribed and inscribed squares (Figure 1), if we take the arithmetic mean of their areas as an approximation of that of the circle, we would obtain the area of the circle $\approx 3r^2$, or $\pi \approx 3$.



$$
\begin{aligned}
\text{Area of Circumscribed Square} &= 4r^2 \\
\text{Area of Inscribed Square} &= 2r^2 \\
\text{Mean} &= \frac{4r^2 + 2r^2}{2} \\
&= 3r^2
\end{aligned}
$$

Figure 1

A group of pre-service teachers were asked to extend this idea to find upper and lower bounds for the sphere, and estimate its volume using the mean of the volumes of the bounds. One of them suggested using the circumscribed cylinder for the upper bound, and a pair of inscribed joined cones for the lower bound (Figure 2). Surprisingly, the mean of the volumes of these upper and lower bounds is $\frac{4}{3}\pi r^3$, which is exactly the volume of the sphere!



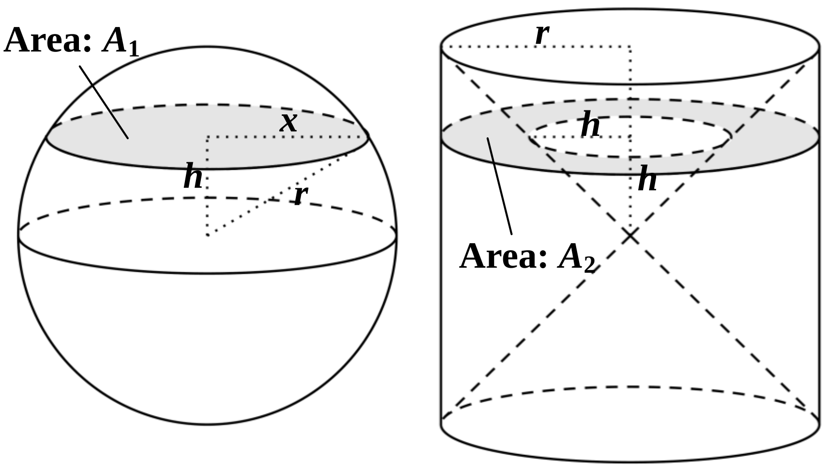
$$\text{Volume of Circumscribed Cylinder} = 2\pi r^3$$

$$\text{Volume of Inscribed Joined Cone} = \frac{2}{3}\pi r^3$$

$$\text{Mean} = \frac{2\pi r^3 + \frac{2}{3}\pi r^3}{2} = \frac{4}{3}\pi r^3$$

Figure 2

This happy coincidence could be further illustrated as follows. Let S, C and B be the volumes of the sphere, its circumscribed cylinder and inscribed joined cones (also called *bicone*) respectively. We invert the two cones and subtract them from the cylinder, as shown in Figure 3. By Pythagoras' Theorem the areas of the shaded cross-sections at any height are equal, and by Cavalieri's Principle the sphere and the cylinder minus the inverted cones have equal volumes, that is, $S = C - B$. Since we also know that the volume of a cone is one-third of that of its circumscribed cylinder, we have $C = 3B$. Hence $\frac{C+B}{2} = 2B = C - B = S$, that is, the mean of the volumes of the circumscribed cylinder and the inscribed joined cones is the volume of the sphere.



$$\begin{aligned} A_1 &= \pi x^2 \\ &= \pi(r^2 - h^2) \\ &= \pi r^2 - \pi h^2 \\ &= A_2 \end{aligned}$$

Figure 3

Finally, it is worthwhile to note that the sphere, cylinder and bicone can be described as the solid of revolution of the *superellipse* $|x|^m + |y|^m = 1$. (http://mathworld.wolfram.com/Superellipse.html) When $m = 2$ it gives the circle, whose surface of revolution is the sphere. When $m = 1$ it gives a pair of straight lines of slope 1, whose surface of revolution is the bicone. When m tends to infinity, we have a pair of horizontal lines which generate the cylinder as the surface of revolution.

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