Understanding students' concepts of slope using diagnostic tasks

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This article reports on a study conducted in 2009 for understanding students' concepts of slope in Key Stage 3. The study was initiated as a result of preliminary analysis of students' performance in the Territory-wide System Assessment (TSA) up to 2008. It was also intended to be the beginning part of a series of studies for improving teaching and learning of secondary mathematics based on in depth analysis of assessment and pedagogy in selected topics. The main part of the study consists of using diagnostic tasks as a means to assess students' learning as well as to suggest ordinary classroom learning tasks for conceptual understanding. It is assumed that teachers can improve pedagogical practice with similar emphasis on task design and better understanding of learning process of individual mathematical concepts.

Background

The introduction to the concept of slope is covered in the learning unit "Coordinate Geometry of Straight Lines". As far as TSA is concerned, there are four Basic Competency (BC) descriptors (KS3-MS13-1 to 4) in this unit (see http://cd1.edb.hkedcity.net/cd/eap_web/bca/eng/BCs/BCs_Math3.htm). The second descriptor (KS3-MS13-2) specifies that students can "use the formula slope = $\frac{y_2 - y_1}{x_2 - x_1}$". The fourth descriptor (KS3-MS13-4) requires students to "demonstrate recognition of the conditions for parallel lines and perpendicular lines". The remaining two descriptors require students to use the two other basic formulas in elementary coordinate geometry: the formulas for calculating distance (KS3-MS13-1) and mid-point (KS3-MS13-3).

These basic formulas are important tools for mathematics learning in the next stage. In particular, the idea of slope is essential for coordinate treatments of geometrical problems on one hand, as well as analyzing linear or even non-linear relations of quantities on the other hand.

TSA papers in the last three years include simple questions for testing students' knowledge about the direct use of the formula of slope. Students' performance in this area remains in a rather low level throughout the years, suggesting that attention is needed to improve their understanding of this basic formula in order to prepare them for more successful learning in the higher-level mathematics.

To investigate how students understand the concept of slope (which should be more than "using the formula" of slope), a set of diagnostics tasks is designed to probe students' understanding in interviews. Based on the results of student interviews, some support materials are later developed to enhance the learning and teaching on the concept.

Analysis of the Topic (Concept of Slope)

The use of the formula for calculating slope is included in a learning unit of elementary coordinate geometry of straight lines. The study of coordinate geometry in this Key Stage 3 provides students with an important experience of learning geometry through an analytic approach (compared with intuitive and deductive approaches). Problems and exercises in this stage usually focus on geometric aspects of the problem situations. Together with the distance formula, students have the basic means to determine length and direction of geometric elements in rectilinear figures represented in a coordinate system.

However, it should also be noted that the use of slope formula is not confined to solving purely geometric problems. It is an essential tool for studying quantitative relations represented in graphical form, connected with understanding of general mathematical functions particularly in the Key Stage 4 and beyond. Currently in Key Stage 3, there is also introduction of graphical methods to study of equations in the dimension of number and algebra. What we notice in the common textbook arrangements is that connection between concept of slope and graphical study of equations is not given attention in Key Stage 3. This is different from other curricula such as those of Mainland China and UK, where formula for slope is also introduced in junior secondary level but mainly as calculation of rate of change in graphical representations of functions. Moreover, students' performance in areas regarding basic technique and knowledge of graphs of equations (in the Number & Algebra dimension) is generally poor according to TSA results.

Besides the study of TSA data, there is a review of common textbook treatments in this topic in order to understand what could probably be the usual approach in teaching. It is found that although explanation of the formula is usually made with respect to some detailed graphical descriptions of points and lines, subsequent examples and exercises seldom refer to any supportive graphical or geometric representations. The emphasis is normally put on the correct computation with the formula using given information of the coordinates while understanding of corresponding graphical or geometric situations is not essential. For example, even when a schematic diagram is provided, it is usually not considered as a true scale drawing and students need not attend to those spatio-graphical details in order to answer the questions. The major question arising is that how far the students can relate the calculation of slope to graphical elements concerned for a sound understanding of the concept.

Design of Diagnostic Tasks

The TSA results indicate clearly students' general weakness in using the slope formula and relating slopes in parallel or perpendicular lines, as we can see in Figure 1 – 2 below. These are simple items and testing mainly the basic knowledge. To further understand students' difficulties in learning this topic, we need more elaborated tools. A set of tasks is designed for this purpose and supplement common approaches in teaching this topic. The main concern is not the process of computing slope based on the formula, but the connection between the slope as a measurement and the graphical/geometric elements that students can directly access.

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If A (3 , 1) and B (−2 , −3) are two points in a rectangular coordinate plane, find the slope of the straight line AB.

		Territory-wide percentage:
A.	$\frac{5}{4}$	*14.6%*
B.	$\frac{4}{5}$	*65.1%*
C.	$-\frac{1}{2}$	*12.6%*
D.	-2	*6.9%*

Figure 1

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The slopes of four straight lines L_1 , L_2 , L_3 and L_4 are given in the following table:

Line	L_1	L_2	L_3	L_4
Slope	5	-5	-5	$-\frac{1}{5}$

Which of the following pairs of straight lines are perpendicular to each other?

		Territory-wide percentage:
A.	L_1 and L_2	*15.1%*
B.	L_1 and L_4	*46.1%*
C.	L_2 and L_3	*34.4%*
D.	L_3 and L_4	*3.8%*

Figure 2

Two groups of secondary 4 students in a high performing school were invited to do the test. Secondary 4 students were selected because we want to understand conceptions of students who have learned this topic for more than one year and are still using this formula in Key Stage 4 mathematics learning. It is assumed that difficulties about concept of slope could also be found in students with higher performance and those students may be more articulate about their thinking in doing the test. The test was first conducted in 2 classes, one in science stream and the other non-science. All their scripts were collected and marked for further selection of students in the next stage of interviews. The results in the science class showed that they could answer most of the questions satisfactorily without obvious pattern of mistakes. Results in the non-science class were different with a variety of unexpected responses. Several students from the non-science class were then identified and invited for individual interviews few days later in order to have them explain how they answered the questions in the test.

There are altogether 8 questions in the test. We illustrate some of them and explain the principles of their settings below.

Question 1 requires students to construct a line parallel or perpendicular to a given line on a square grid (Figure 3). There is no direct reference to any coordinate system. It is intended to test students' geometric conception of inclination of lines without consideration of slope.

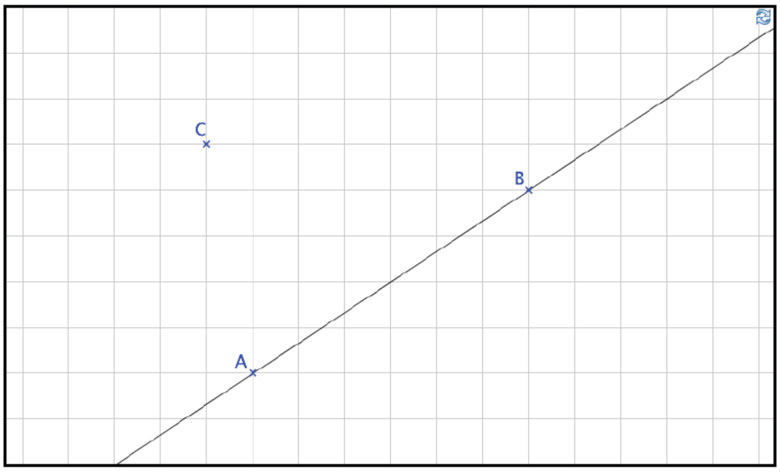


Figure 3

Question 2 requires students to construct lines with specific slopes (Figure 4). This is an open question for the lines can be drawn in any position inside the graph and different elements of the coordinate system or square grid can be made use of for their constructions. This question allows a student to show what a slope measurement mean for a straight line in the coordinate plane.

Question 2: In figure 2,

- draw a line with slope 1;
- draw a line with slope $\frac{1}{2}$.

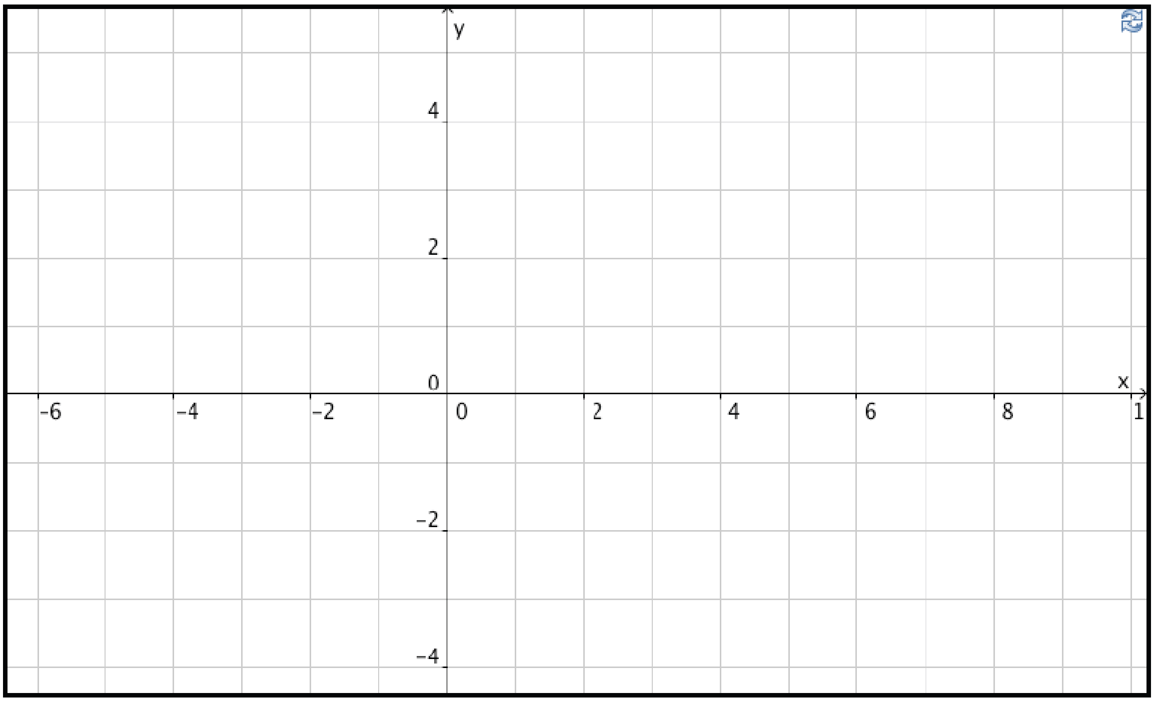


Figure 4

Question 3 requires students to calculate or estimate slopes of lines in simple diagrams drawn to scale (Figure 5). The main idea is that no coordinates of points on the line are directly given but they can choose convenient points for the purpose of determining the slope. However, actual calculation with the coordinates is not necessary for if they realize that slope is basically a ratio about changes in the coordinates (variables), the changes can be directly read from the graph, instead of individual coordinates.

Question 4 asks students to suggest points forming segments of various slopes in the graph (Figure 6). Once again, detailed calculations may not be needed if they make use of the square grid and reasoning about horizontal, vertical changes and ratios.

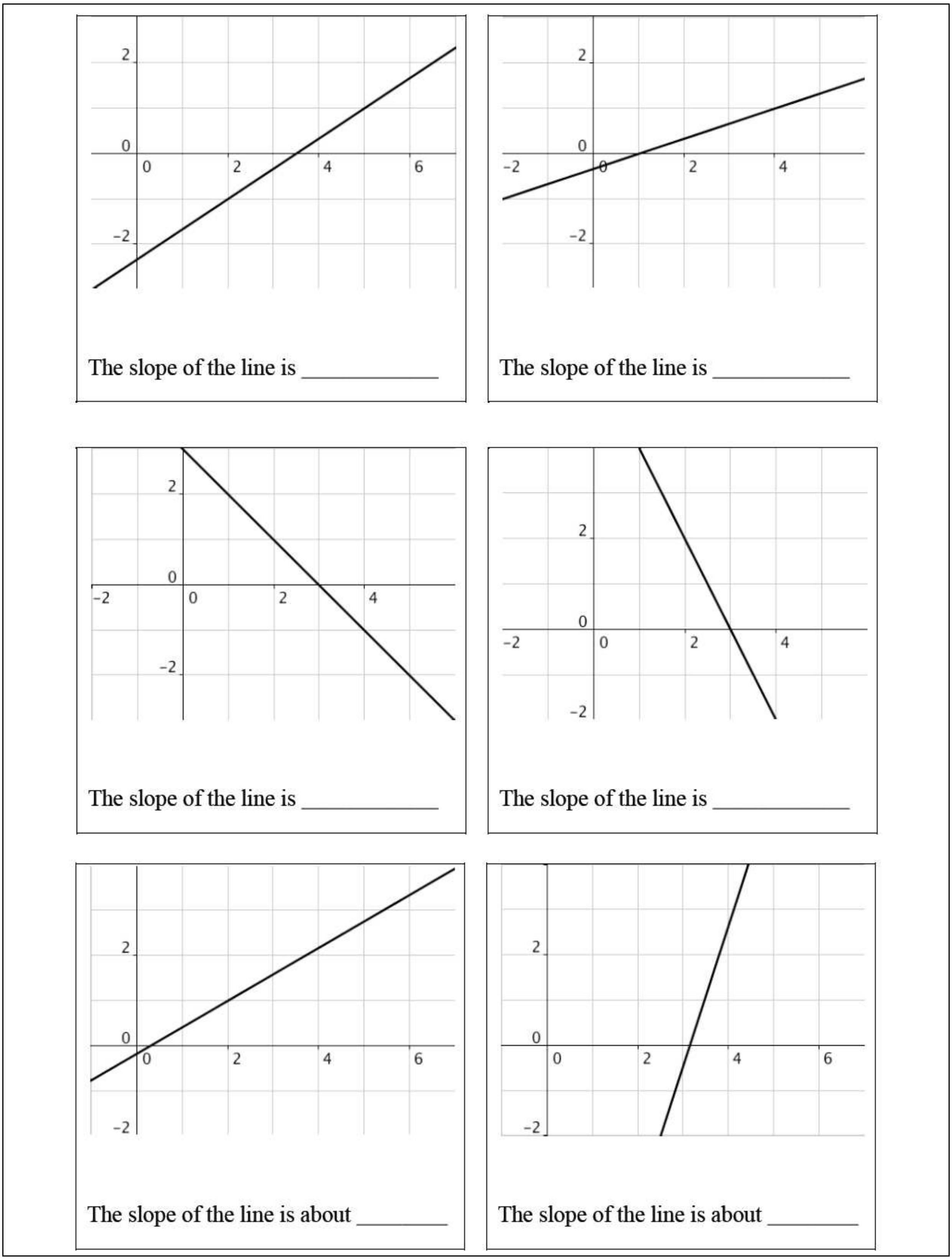

The slope of the line is ____________
The slope of the line is ____________
The slope of the line is ____________
The slope of the line is ____________
The slope of the line is about ________
The slope of the line is about ________

Figure 5

Question 4: In the figure 4, the coordinates of A are (3,1). Add some points to the figure with the following requirements.

	Coordinates of the point you choose
Find a point B so that the slope of AB is $\frac{1}{2}$.	B(,)
Find a point C so that the slope of AC is $\frac{4}{3}$.	C(,)
Find a point D so that the slope of AD is 2.	D(,)
Find a point E so that the slope of AE is -3.	E(,)
Find a point F so that the slope of AF is $-\frac{1}{3}$.	F(,)

In figure 4, draw the points B, C, D, E, F chosen above.

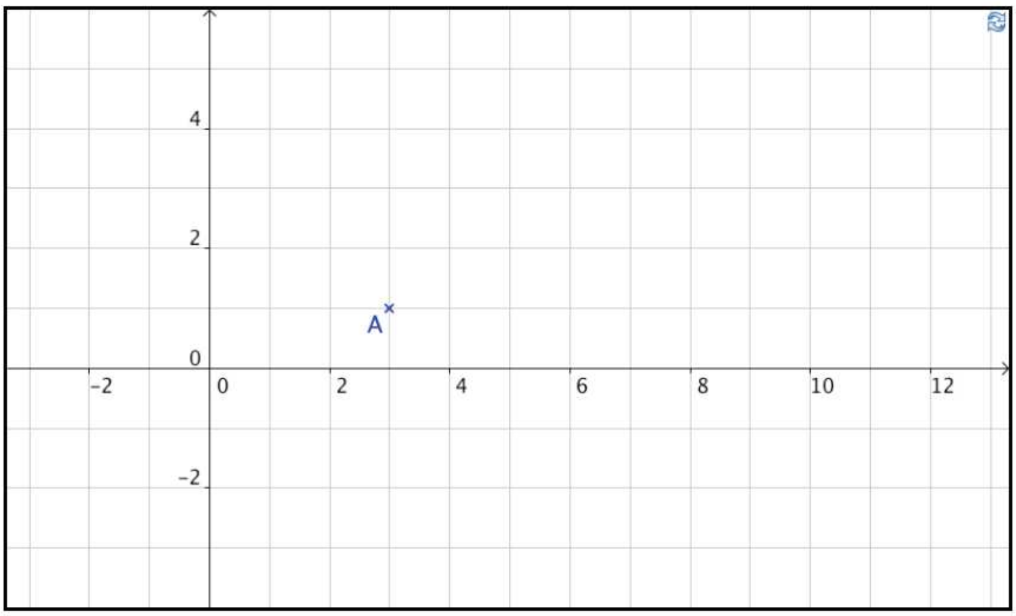


Figure 6

Similarly, question 5 asks students to provide points to form segments but with the same slope (Figure 7). It is an important concept for students to realize that the value of slope does not depend on the choice of points on the same line.

Question 5: In the figure 5, the coordinates of A are (3,1). Add some points to the figure with the following requirements.

	Coordinates of the point you choose
Find a point B so that the slope of AB is $\frac{2}{3}$.	B(,)
Find another point C, if possible, so that the slope of AC is also $\frac{2}{3}$.	C(,)
Find another point D, if possible, so that the slope of AD is also $\frac{2}{3}$.	D(,)
Find another point E, if possible, so that the slope of AE is also $\frac{2}{3}$.	E(,)

In figure 5, draw the points B, C, D, E chosen above.

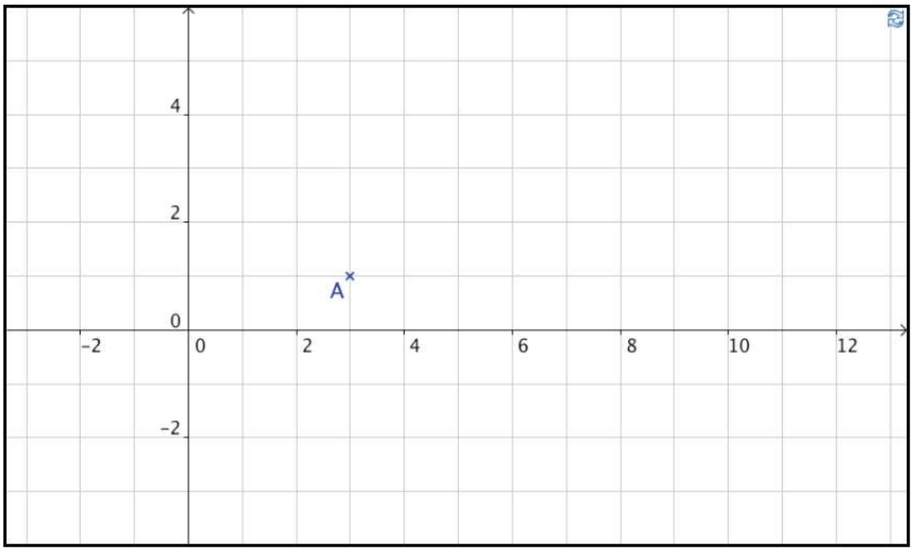


Figure 5

Besides the points chosen above, will there be other points that can make a segment of slope $\frac{2}{3}$ with point A? Explain your answer in the box below.

YES / NO

Figure 7

Questions 6 to 8 test students' understanding and technique of relating coordinates (as well as directed numbers) on a single number line, which is essential to successful calculation of slope and describing changes in values within a variable.

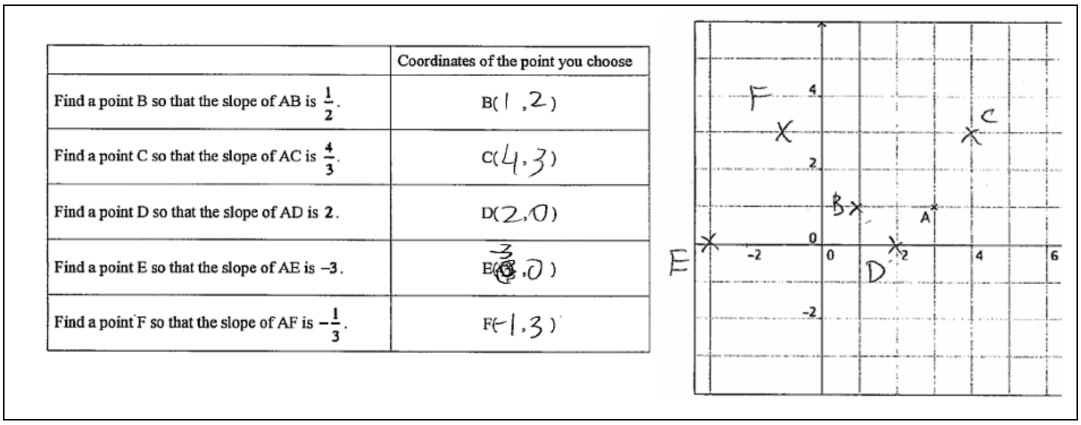
Findings from Interviews

In the chosen school, students from the non-science class indicated a range of misconceptions about slope of straight lines both in the written test and interviews. The video-recorded interviews provide valuable information for analyzing students' reasoning and understanding of the meaning of slope. Although there were serious misconceptions reflected in their work and explanation, basically their computational skills were not poor and some might correctly calculate with the formula with ease. The interviews were conducted in Cantonese. In the interviews, we started with question 4 and 5, which showed a variety of unexpected answers from the test. Interviewees were asked to work on these questions directly (without referring to their previous answers in the written test) and immediately explain their methods and answer other questions from the interviewers. After finishing the questions 4 and 5, some were asked to return to question 1 (if time allows) to simply construct parallel and perpendicular lines.

Major findings from the interviews are listed below.

Some students interpret the value of slope (expressed as a ratio of 2 numbers) as a pair of coordinates, instead of changes in coordinates. This is noted in some students' answers in question 4 (Figure 8). Some students have difficulties in forming segments with specific slopes because they unnecessarily assumed that the line must start from origin when talking about slope.

Students' interpretation of slope or related geometric properties may too often be affected by prototypical images (Presmeg, 1997). For example, lines passing through origin may be convenient cases for illustrating certain slopes and students' images may then be confined to lines of this characteristic. Another way to understand this difficulty is that students' experience in handling different examples of geometric objects is very limited.



	Coordinates of the point you choose
Find a point B so that the slope of AB is $\frac{1}{2}$.	B(1 ,2)
Find a point C so that the slope of AC is $\frac{4}{3}$.	C(4,3)
Find a point D so that the slope of AD is 2.	D(2,0)
Find a point E so that the slope of AE is −3.	E(-3,0)
Find a point F so that the slope of AF is $-\frac{1}{3}$.	F(-1,3)

Figure 8

Use of formula itself for calculating slope (once specific coordinates are identified) is not difficult for these students, but the understanding of the geometric relations shown in the diagrams (which are drawn accurately) is very weak. For example, in Question 3 students could only find the slope of each line by identifying the coordinates of two points on the line and using the slope formula (Figure 9). None of these students try to find the slope by finding the vertical and horizontal changes (by directly counting squares on the grid) and their ratio.

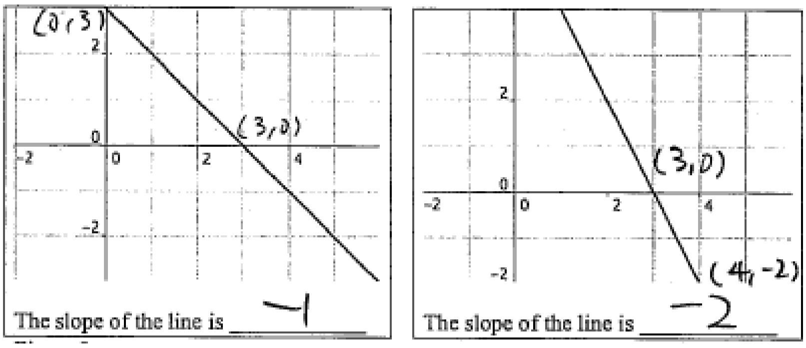


Figure 9

Some students cannot understand how value of slope may change with inclination of a line. Some can distinguish lines with positive and negative slope but cannot explain or describe how they are different. It is surprising to know that some students' interpretation of slope depend on the nature or form of the slope value. For example, a student interprets a line of slope ½ as an inclined line, while a line of slope 2 is a horizontal line with *y*-intercept 2.

When students come to construct parallel and perpendicular lines without explicitly considering slope in Question 1, they have difficulties in drawing these lines. Although many students know (but are unable to explain) that parallel lines have equal slopes, they do not make use of this relation to draw the parallel line. Many of them just draw the parallel line by translating the ruler (Figure 10). It seems that their considerations of parallel and perpendicular lines are not connected with the idea of slope even when square grid is clearly shown.

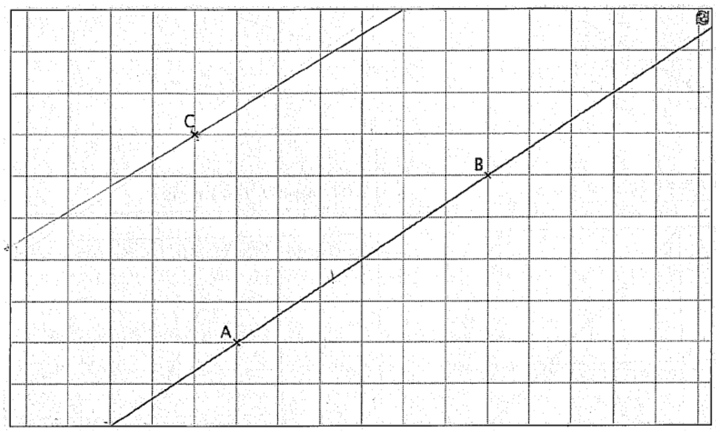


Figure 10

On the other hand, some students note that the slope can be considered but have to set up a very detailed coordinate system on the square grid and rely on the calculation based on the coordinates obtained (Figure 11) instead of directly counting squares on the grid and check the simple ratios.

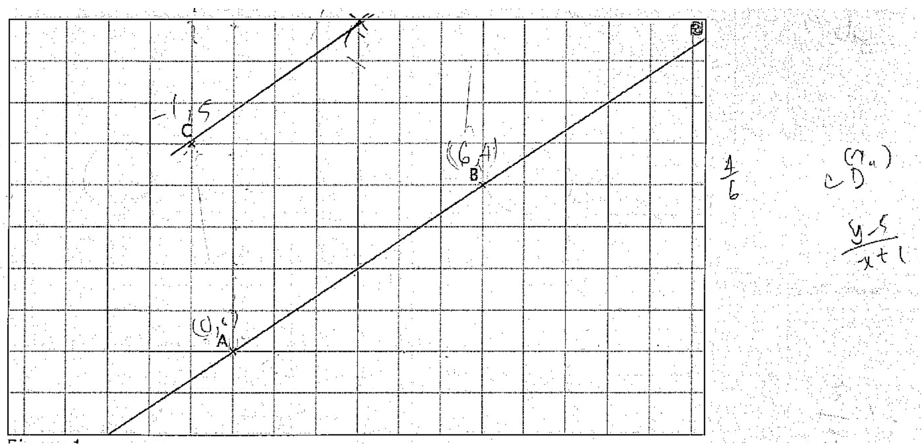


Figure 11

There are different kinds of problems handling perpendicular lines. Some cannot construct perpendicular lines on square grid (Figure 12). Some mix up the terms "perpendicular" and "vertical". Relation between slopes of perpendicular lines are also confusing with some assume that they have equal slopes.

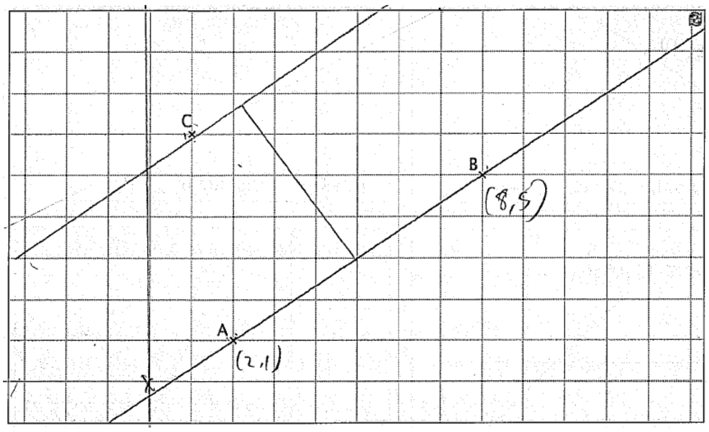


Figure 12

Discussion and Suggestions

Although the sample for this study is small, the discovery of various misconceptions of students in this topic suggests possible direction for improving students' conceptual understanding through our teaching. We see students failing to interpret geometric representations of lines related to their slopes even when they can basically master the calculations with the slope formula. It is reasonable to expect similar difficulties in other students of Key Stage 3 who cannot even use the formula properly. Moreover, the usual teaching approaches suggested by the textbooks seldom provide continuous support for enriching the connection between geometric representations of straight lines and calculated slopes once the formula is derived.

Based on the findings from interviews, we have the following suggestions to improve students' conceptual understanding on slope.

1. Discuss with students how to compare the "steepness" of inclined planes of different heights and widths (Figure 13). After discussions, formulate with students the ratio "height / width" as a measurement of the steepness of inclined planes.

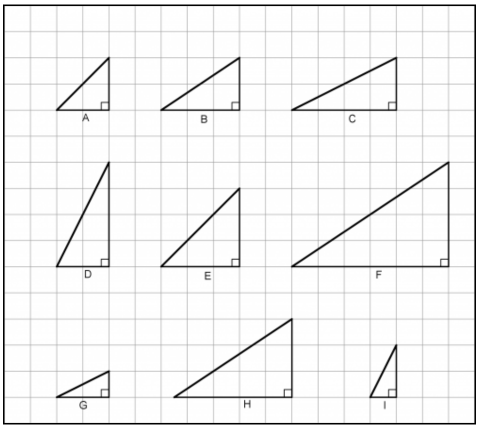
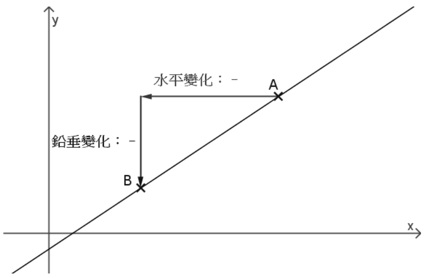


Figure 13

2. In Cartesian plane, introduce the terms "vertical change" and "horizontal change" from point A to point B and the notions of their signs (Figure 14). Slope is defined as the ratio "vertical change / horizontal change". Students are then asked to calculate slopes by directly counting squares on the grid (Figure 15).



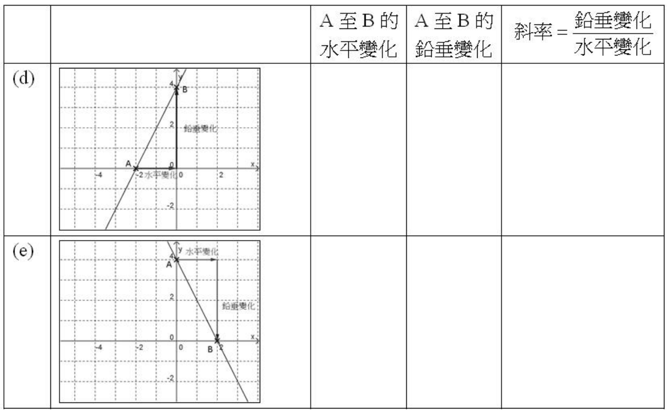


Figure 14

Figure 15

3. Based on previous discussion, introduce the slope formula. To avoid mixing up the orders of x_1, x_2 and y_1, y_2, one can consider introducing the "vertical form" of the formula as shown in Figure 16.

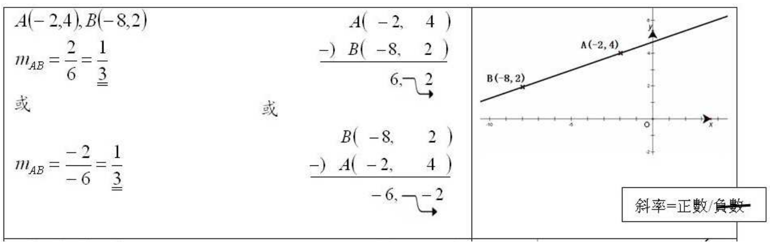


Figure 16

4. To enhance students' understanding on the connection between the value of slope and the geometric representation of the straight line, dynamic geometry software could be used to demonstrate how the straight line and the slope vary if one fixes the horizontal distance between two points but increases the vertical (Figure 17), or fixes the vertical distance and increases the horizontal.

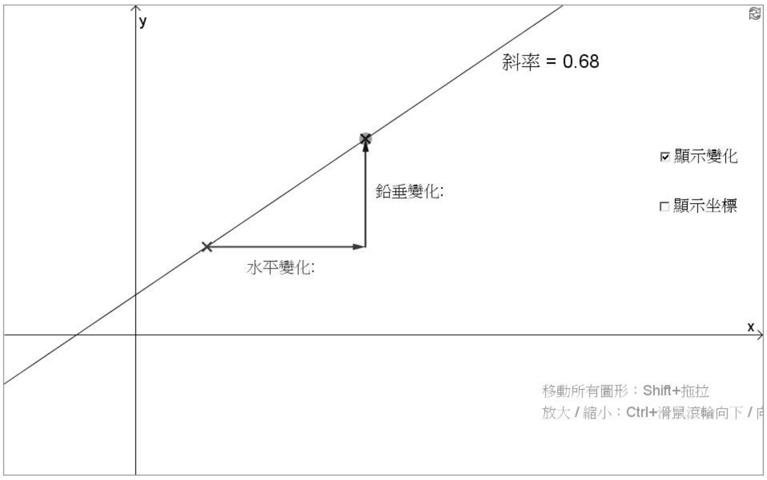


Figure 17

5. To facilitate learning the conditions for parallel lines and perpendicular lines by understanding instead of by rote, students could first be asked to investigate the strategies of drawing lines parallel and perpendicular to a given line ***using the square grid***. It is then possible to formulate the conditions of their slopes according to these strategies. Again, dynamic geometry software would be a useful tool to support the investigation (Figure 18 and 19).

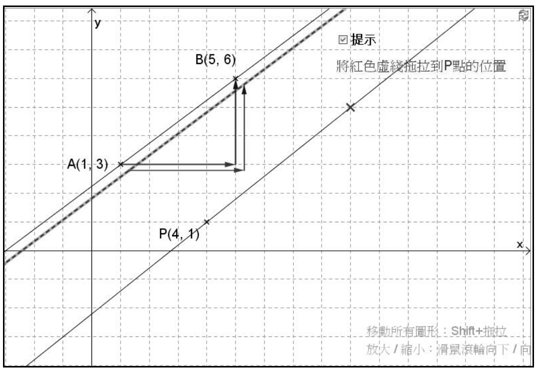


Figure 18

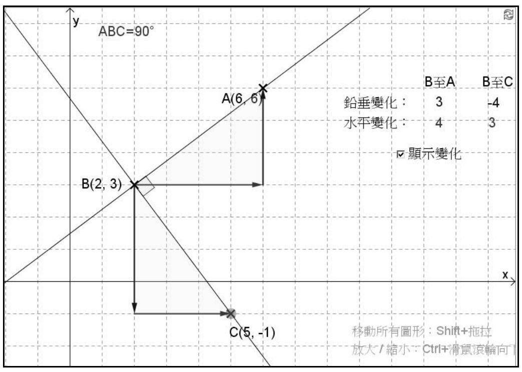


Figure 19

Figures 11 – 19 are extracted from the Web-based Learning and Teaching Support (WLTS) developed by Education Bureau. Interested readers can download the whole set of materials on this topic (only Chinese version available at the moment) from either the EdBlog (教城網誌) of WLTS (http://edblog.hkedcity.net/math-wlts/2010/01/22) or its official website (http://cd1.edb.hkedcity.net/cd/eap_web/bcalt/tr/index.htm).

This set of materials was tried out in another school with a secondary three class. Although the students in this school are relatively low achieving, preliminary evaluation with post lesson tests indicates that effect of the suggested approach is positive. Students in general have more comprehensive understanding of the concept of slope. In post-lesson interviews, some students were able to explain the sign of the slope from the geometric representation of a line by considering the horizontal and vertical changes between two points on the line. One student could even explain the condition for perpendicular lines using rotation! It is also notable that many weak students found the "vertical form" of the slope formula very helpful, while the more able students preferred the original form which is more "convenient" for them.

The various misconceptions discovered in this study may be the results of the common treatment in this topic which emphasize on the use of the formula for calculating slope. In fact it is required in the curriculum guide to "understand and use formulas of distance and slope" and to "understand the conditions for parallel lines and perpendicular lines" (Curriculum Development Council 1999, p.24). As mathematics teachers we all support learning by understanding. The question is "How?" This article reports on a case how diagnostic assessment tools could be used to inform us the learning needs of students, and bring us insights for designing appropriate teaching strategies and materials for improving their understanding in the topic. It is hoped that this case could suggest a possible direction for us to enrich our understanding of the learning process and the pedagogical practice of individual mathematical concepts.

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Presmeg, N.C. (1997). Generalization using Imagery in Mathematics. In L.D. English (Ed.) *Mathematical Reasoning: Analogies, Metaphors, and Images*. London: Lawrence Erlbaum Associates.

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