Bisecting a Trapezium

Francis Lopez-Real
The University of Hong Kong

A Construction Problem

As part of the PGDE Mathematics Major course at the University of Hong Kong we devote a number of sessions to problem solving. In some of these sessions students are given a range of problems to solve where the purpose is not simply to find a solution but also to identify the problem-solving strategies they use, to see if alternative solutions are possible, and to consider how helpful the use of ICT (Information and Communication Technology) may be. In particular, geometry problems are usually rich in alternative solutions and I discuss one such problem here:

Given any point P on the shorter parallel side of a trapezium, construct a line that bisects the trapezium into two equal areas.

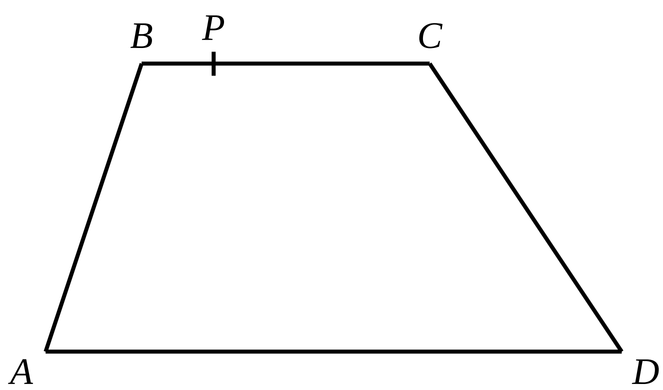

Figure 1

The most common solution produced (perhaps unsurprisingly) is as follows.

SOLUTION 1 In Figure 2, join MN (the mid-points of BC and AD respectively). Mark E, the mid-point of MN. Construct PQ through E as shown.

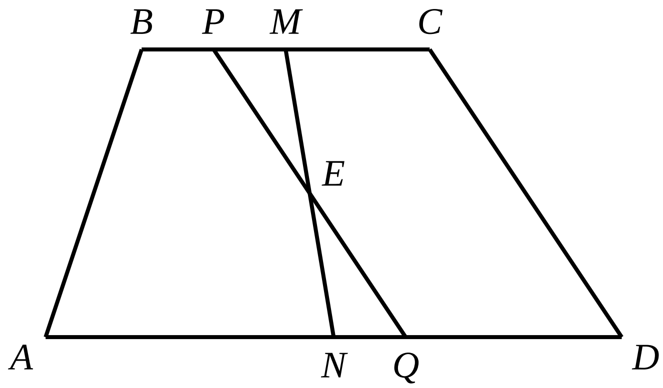

Figure 2

This construction is based on the strategy of first identifying a known line that does bisect the trapezium. In this case, MN is clearly such a line. Then we require a line from P, crossing the line MN at E and intersecting AD at Q, such that $\text{area}(\Delta PME) = \text{area}(\Delta QNE)$. (This is a "balancing" strategy since $\text{area}(\Delta PME)$ is being subtracted from the area of $ABMN$ and $\text{area}(\Delta QNE)$ is being added to it). It is easy to show that if E is taken as the mid-point of MN then our aim is achieved (since ΔPME and ΔQNE are then in fact congruent). In addition to this construction some students found the following two solutions.

SOLUTION 2 In Figure 3, mark M and N, the mid-points of AB and CD respectively. Draw the lines PMG and PNH. Mark Q, the mid-point of GH. Then PQ bisects the trapezium $ABCD$.

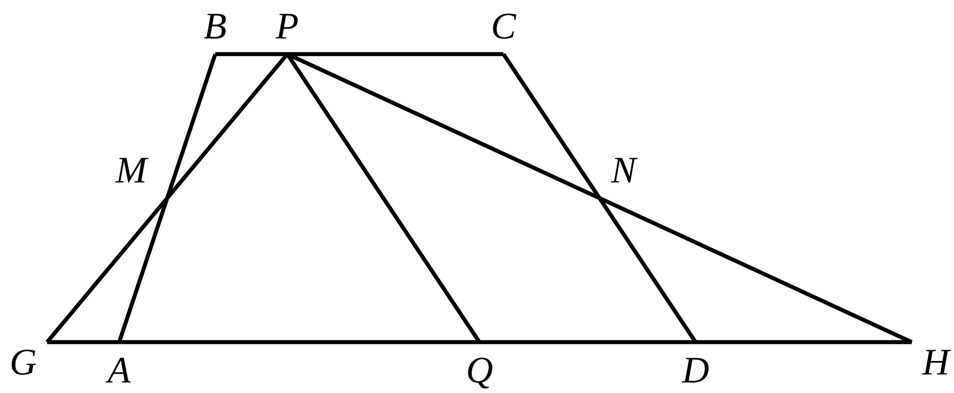

Figure 3

The strategy in this construction is to replace $ABCD$ by a triangle of equal area, with P as its apex (since we know that the area of a triangle is easily bisected by a median). Again, it is fairly trivial to see that choosing the mid-points M and N will achieve this aim. (As with the previous construction, this can also be interpreted in terms of "balancing" areas). A third solution produced by the students appears slightly indirect but is quite ingenious:

SOLUTION 3 In Figure 4, construct $TD = BP$ and $AS = PC$. Join PS and PT. Mark Q, the mid-point of ST. Then PQ bisects the trapezium $ABCD$.

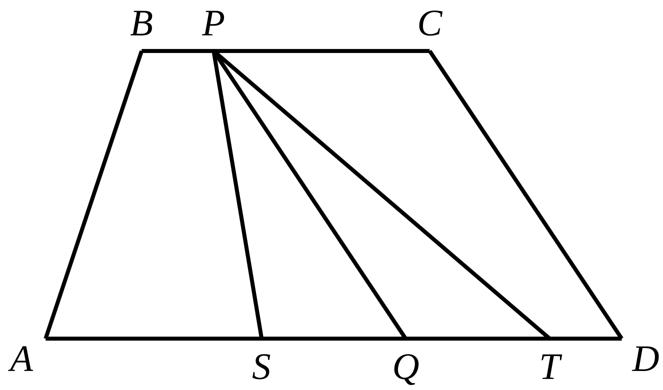

Figure 4

This construction again seeks to effectively reduce the problem to bisecting a triangle. But here, two equal area trapeziums ($ABPS$ and $DCPT$) are first constructed on either side of P. Having divided the trapezium into three areas, the outer two of which are equal, the problem is reduced to bisecting the middle area (i.e. ΔPST).

I am quite sure other constructions are possible and the reader might like to try to find yet another solution. However, apart from the challenge of finding alternative solutions, such a problem also suggests other related problems that could be tackled. For example, if we consider triangles, can we find a construction that bisects any triangle from any point on one of its sides? What about any quadrilateral, or any polygon? Clearly the latter examples are likely to be very challenging indeed at secondary level, but let's consider the first of these suggestions.

Bisecting Triangles

The problem can be described by reference to Figure 5.

Given any ΔABC *and a point* P *on one side (say* BC *), construct a line* PQ *that divides* ΔABC *into two equal areas.*

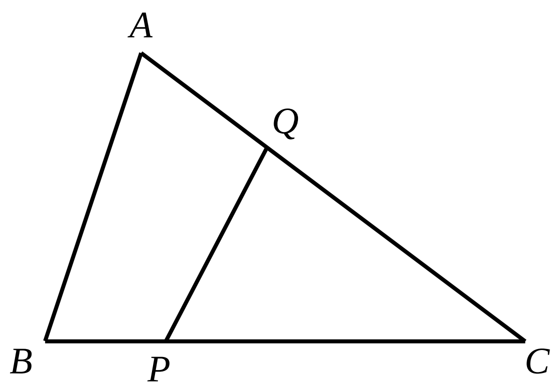

Figure 5

If we use a similar strategy to that described in SOLUTION 1 of the trapezium problem, we could start by identifying a known line (drawn to BC) that does bisect the triangle. The obvious choice is the median AM , as shown in Figure 6a.

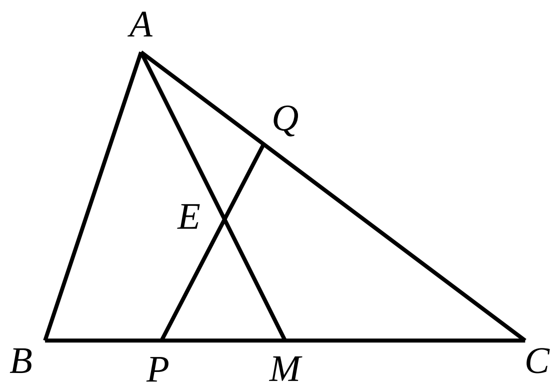

Figure 6a

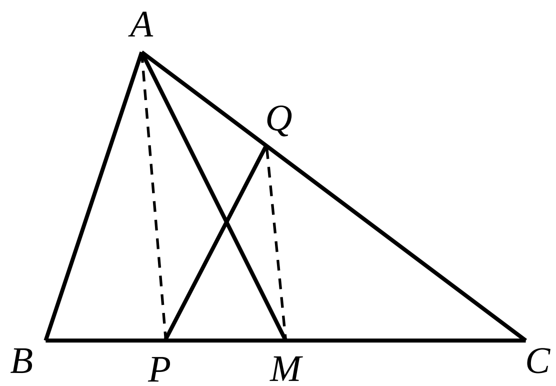

Figure 6b

We need to construct a line PQ , crossing AM at E , such that area(ΔPEM) = area(ΔAEQ) (using the "balancing" principle again). But if we join AP then this is the same as requiring area(ΔAPM)= area(ΔAPQ). This is easily achieved using the theorem that triangles on the same base and between the same parallels are equal in area. Thus we now have quite a simple construction, as illustrated in Figure 6b: Draw the line AP . From M , the mid-point of BC, construct MQ parallel to AP. Then PQ bisects ΔABC. It's worth pointing out here that the construction is valid whether P is to the left of M or the right of M. In the latter case the point Q will then be on AB as shown in Figure 6c.

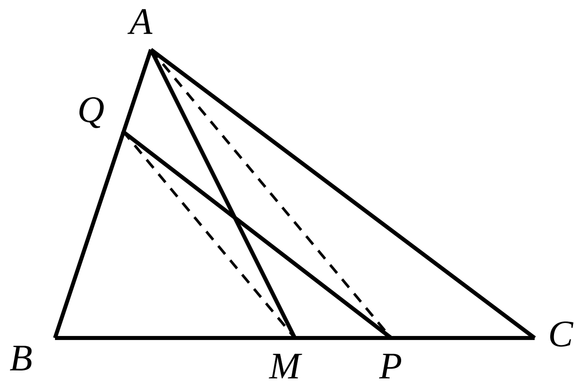

Figure 6c

Back to the Trapezium

Well, the triangle problem turned out not to be as difficult as we might have thought, even though it is more general in nature than the trapezium problem. But notice that we have used similar strategies to those we employed in the trapezium problem. Let's now return to that problem but give ourselves a further challenge. In the original problem it was specified that P should be on the shorter of the two parallel sides. What is the significance of this? In other words, how is the problem changed by starting with any point P on the *longer* parallel side?

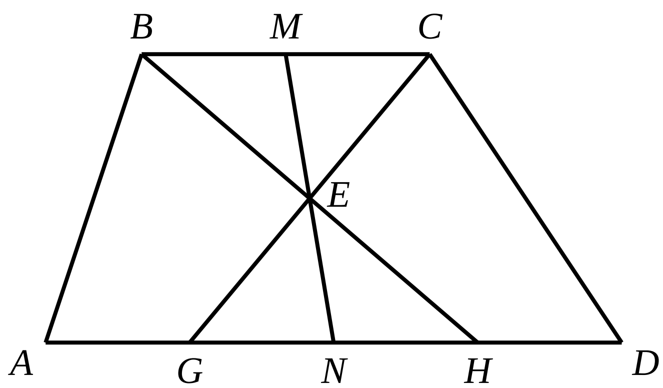

Figure 7

If we consider the first solution to the original problem, we can easily find the range of positions of point Q on AD as point P moves along the whole of BC. This is shown as the line segment GH in Figure 7.

Now consider starting with any point on AD. Clearly if the point lies on GH then the original construction is still valid. But what happens if the point lies within AG or HD? Using the same construction we would have a situation like that shown in Figure 8.

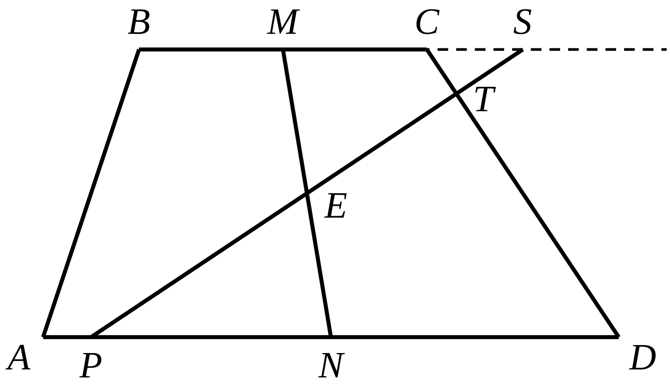

Figure 8

As before, $ABMN$ is half the area of $ABCD$, and ΔPNE is congruent to ΔSME. Hence the construction line PET gives us an area $ABCTP$ which is *less than* half the area of $ABCD$ (by an amount equal to area(ΔCST)). Correspondingly, area(ΔPDT) is more than half of $ABCD$ by the same amount. So the problem now is to construct a point Q on TD such that area(ΔPTQ) = area(ΔCST). It's not immediately obvious how this might be done so let's take it in stages and try to transform ΔCST while keeping its area invariant. In the following diagrams, to avoid confusion, the line MEN is omitted but remember that line PTS was constructed through the point E. (Also, the length of BC has been shortened simply because this makes the subsequent constructions visually clearer).

In Figure 9a, construct $TK = TS$ and join KC. Hence area(ΔCKT) = area(ΔCST). Also, if we join CP we can see that $\dfrac{\text{area}(\Delta CKT)}{\text{area}(\Delta CPT)} =$

$\dfrac{TK}{TP}$.

Now in Figure 9b, construct $TL = TC$ and join PL . Hence area(ΔPCT) = area(ΔPLT).

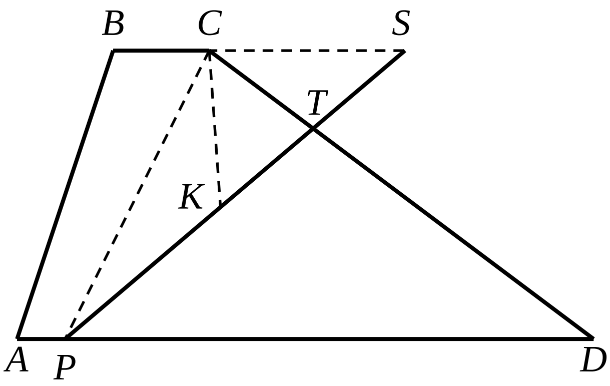

Figure 9a

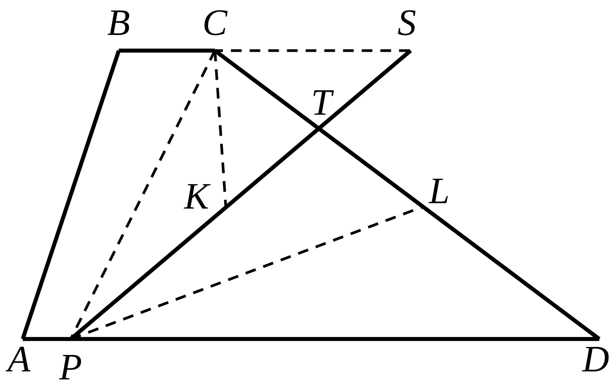

Figure 9b

The problem will now be solved if we can construct Q such that $\dfrac{TQ}{TL} = \dfrac{TK}{TP}$. But this can now be easily done by constructing KQ parallel to PL . This final construction is shown in Figure 9c.

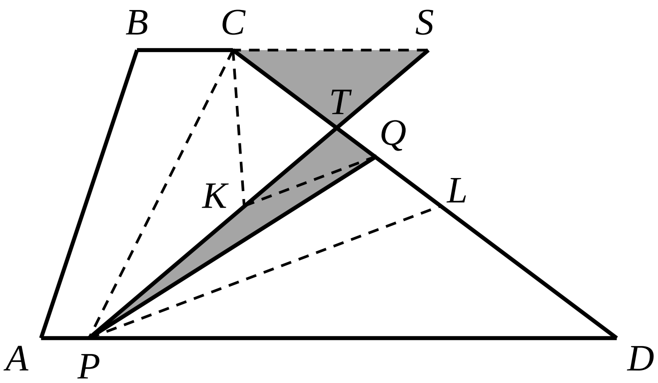

Figure 9c

The two shaded triangles in Figure 9c are equal in area and hence we have achieved our goal of transforming the area of ΔCST to its required new position in the trapezium.

Conclusion

Changing the initial conditions of the trapezium problem certainly made it a great deal more challenging. Even with the help of dynamic geometry software I still found this a tough problem. Apart from my own struggle, its difficulty was made abundantly clear to me by the fact that none of my students was able to solve it. And even after having found a possible construction, I'm less than enamoured by it, to say the least. It's not at all the elegant construction one would hope to arrive at. Perhaps readers may be able to come up with something a little more "direct" and satisfying. As far as other challenges are concerned, I have already suggested a couple earlier. In addition, having seen that bisecting an arbitrary triangle turned out to have quite a neat solution, we might ask ourselves about bisecting the *perimeter* of any triangle from any point on a side. In fact, this is not likely to be a very difficult problem but how about *combining* these two ideas? That is, can we find a line (or more than one) that bisects both the area and perimeter of an arbitrary triangle? It's clear that some special cases exist; the situation for an isosceles triangle, for example, is trivial. However, I suspect that in general this is quite a challenging problem. Again, perhaps readers may like to follow it up.

Author's e-mail: lopezfj@hkucc.hku.hk