Various Methods of Forming a Conjecture for the Number of Diagonals in an *n*-sided Convex Polygon

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1. Introduction

The purpose of this note is to draw attention to some methods of forming a conjecture for the number of diagonals in an *n*-sided convex polygon. This note also highlights some creative works by students in response to a class assignment (see Appendix 1). The assignment was in part a response to a student's question on how one go about forming the formulae that she had to prove in her drill on mathematical induction. The assignment was modified from an example found in Cirrito (2004, pp. 436–437) and it was divided into two parts with the first being a routine drill involving binomial expansion for integer $n \geq 0$. The second part was a small investigation that requires students to generate the correct data using sketches of convex polygons, proposed a conjecture with the help of graphing display calculator, and carry out a proof using mathematical induction.

This note will only focus on the second part of this assignment that concerns the number of diagonals in an *n*-sided convex polygon. The method involves in this proof is also of interest because it departs from the standard examples and drills found in most textbooks on mathematical induction. In this case, students are required to come up with another expression for the number of diagonals in an *n*-sided convex polygon, one other than that obtained from regression, before the proof using mathematical induction can be completed. Thus, in effect, they are required to construct the same conjecture using regression method and a different method of their choice. The later caveat

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represents both a challenge and an opportunity for mathematical creativity for the students.

2. Methods from Students

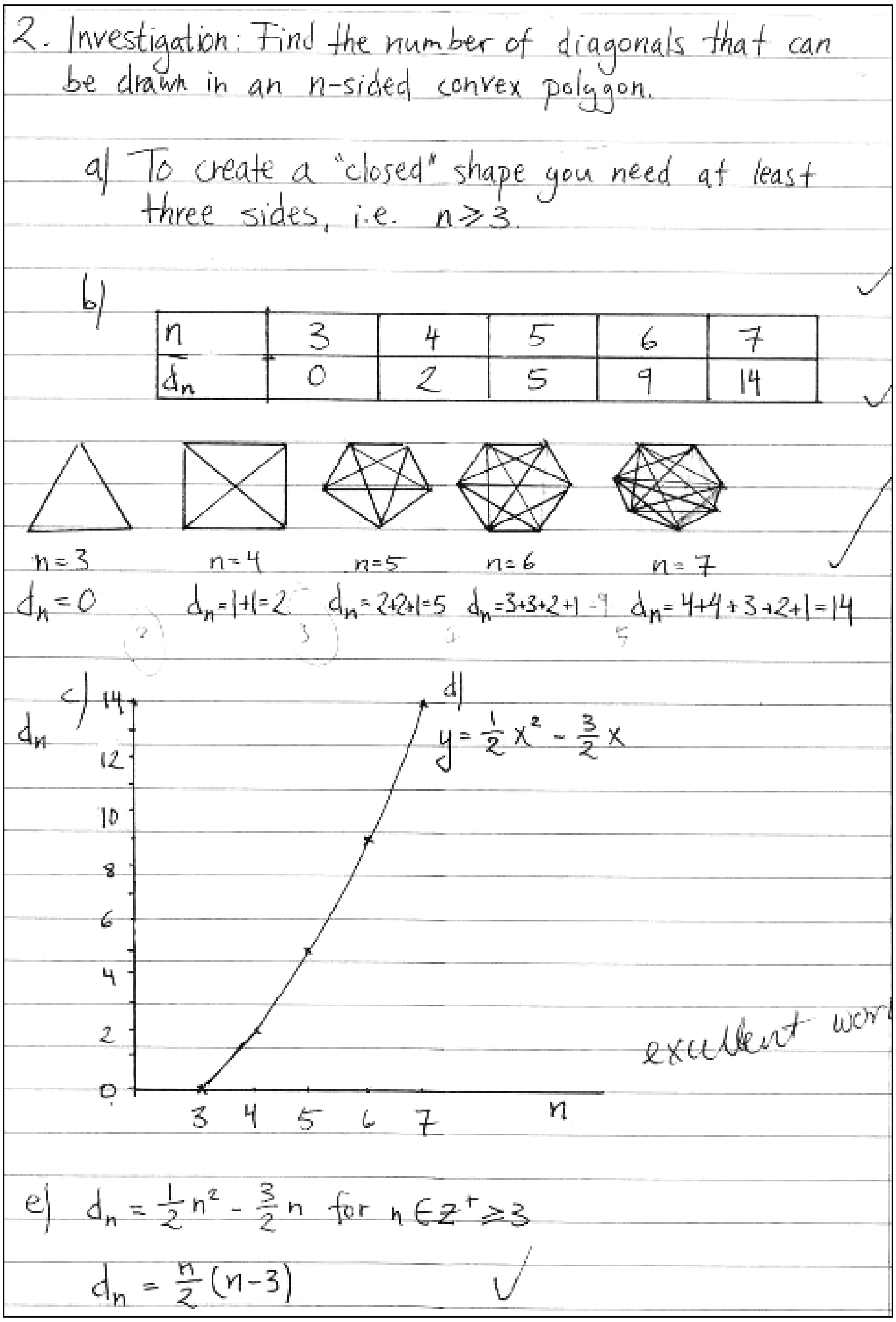


Exhibit 1

Given the leading questions many students in the class were able to generate the correct data, proposed a correct conjecture and identified the limitation in their conjectures. Student A's work (see exhibit 1) is a good representation of work from this class. Students were able to employ their graphing display calculators to fit an appropriate regression model to their self-generated data. Student A also provided a sketch of his regression graph but failed to report that the equation $y = \frac{1}{2}x^2 - \frac{3}{2}x$ came from a regression function in the calculator. The use of symbol x and y in the sketch which was at odd with the names in the axes could be an attempt to indicate that the above equation was obtained from a graphing display calculator.

In order to carry out the proof using mathematical induction, students are expected to use the concept of recurrence equations as shown by Student B's work in exhibit 2. Students had been given the relevant page numbers from Cirrito (2004) as a reference. Many students actually preferred to be creative and appeared to have made little reference to their textbooks edited by Cirrito. Student B was probably the exception. However, her recurrence equation was different from that found in Cirrito (2004, p. 437). Despite the imperfect set-up of her proof, Student B's recurrence equation was correct and reported the appropriate limitation. She did not provide a justification for her recurrence equation. When asked for a justification, she replied that simple inspection of the table was adequate to reveal the recurrence equation.

Here is a formalization of that inspection. Given that

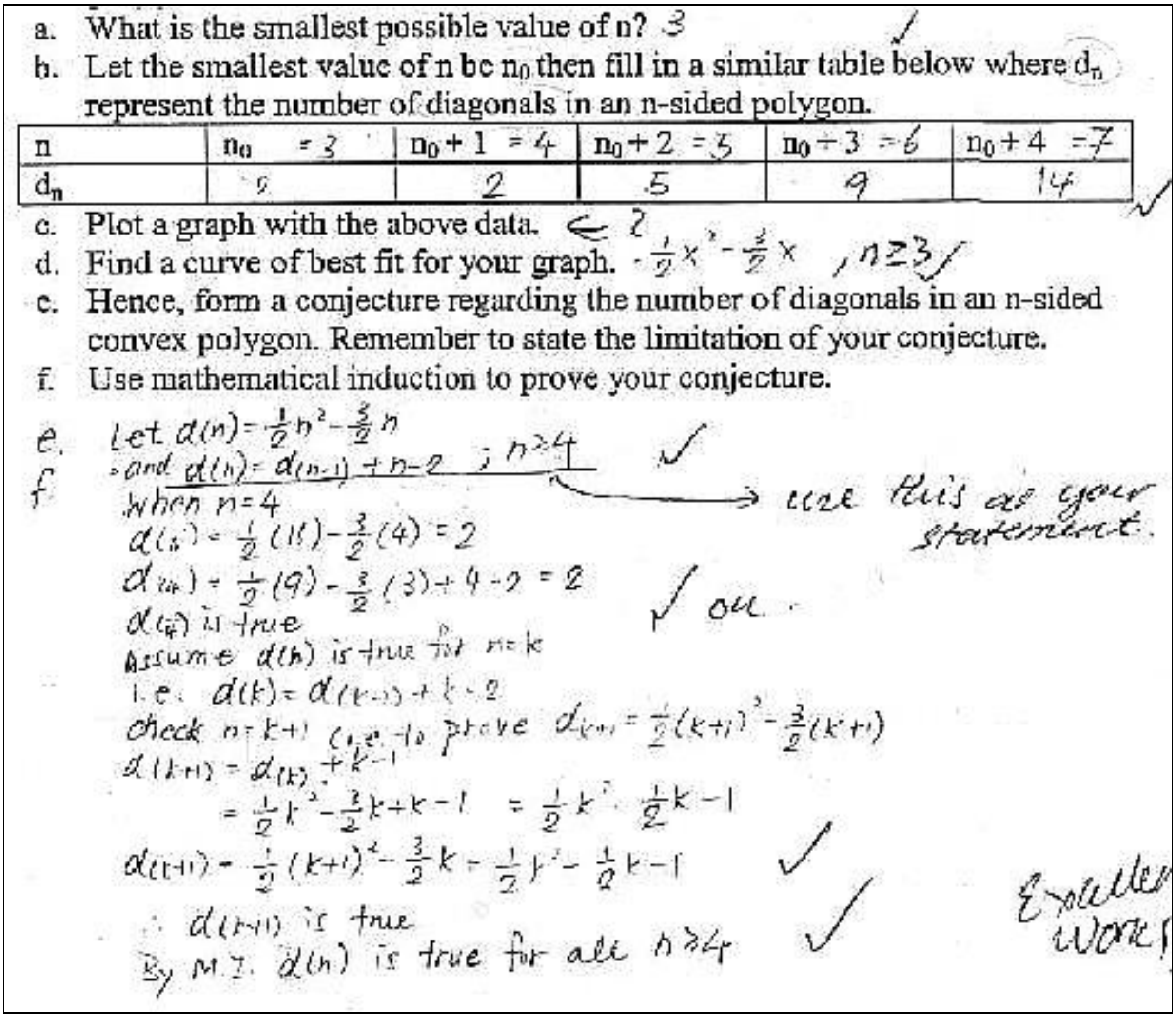
$$d_n = \frac{n(n-3)}{2} \; ; \; n \geq 3 \quad \text{.......................................} (1)$$

we can obtain

$$\begin{aligned} d_{n-1} &= \frac{(n-1)(n-1-3)}{2} \; ; \; n \geq 4 \\ &= \frac{1}{2}(n^2 - 3n - 2n + 4) \; ; \; n \geq 4 \\ &= \frac{1}{2}(n^2 - 3n) - \frac{1}{2}(2n - 4) \; ; \; n \geq 4 \\ &= d_n - (n-2) \; ; \; n \geq 4 \end{aligned}$$

Thus, we have

$$d_n = d_{n-1} + (n-2) \; ; \; n \geq 4 \quad \text{................................} (2)$$



a. What is the smallest possible value of n? 3 ✓

b. Let the smallest value of n be n_0 then fill in a similar table below where d_n represent the number of diagonals in an n-sided polygon.

n	n_0 = 3	$n_0 + 1$ = 4	$n_0 + 2$ = 5	$n_0 + 3$ = 6	$n_0 + 4$ = 7
d_n	0	2	5	9	14

c. Plot a graph with the above data.

d. Find a curve of best fit for your graph. $\frac{1}{2}x^2 - \frac{3}{2}x$, $n \geq 3$

e. Hence, form a conjecture regarding the number of diagonals in an n-sided convex polygon. Remember to state the limitation of your conjecture.

f. Use mathematical induction to prove your conjecture.

e. Let $d(n) = \frac{1}{2}n^2 - \frac{3}{2}n$

f. and $d(n) = d(n-1) + n - 2$; $n \geq 4$ ✓ → use this as your statement.

When n = 4

$d(4) = \frac{1}{2}(16) - \frac{3}{2}(4) = 2$

$d(4) = \frac{1}{2}(9) - \frac{3}{2}(3) + 4 - 2 = 2$ ✓ ok.

d(4) is true

Assume d(n) is true for n = k

i.e. $d(k) = d(k-1) + k - 2$

check n = k+1 (i.e. to prove $d_{(k+1)} = \frac{1}{2}(k+1)^2 - \frac{3}{2}(k+1)$)

$d(k+1) = d(k) + k - 1$

$= \frac{1}{2}k^2 - \frac{3}{2}k + k - 1 = \frac{1}{2}k^2 [illegible] \frac{1}{2}k - 1$

$d(k+1) = \frac{1}{2}(k+1)^2 - \frac{3}{2}k = \frac{1}{2}k^2 - \frac{1}{2}k - 1$ ✓

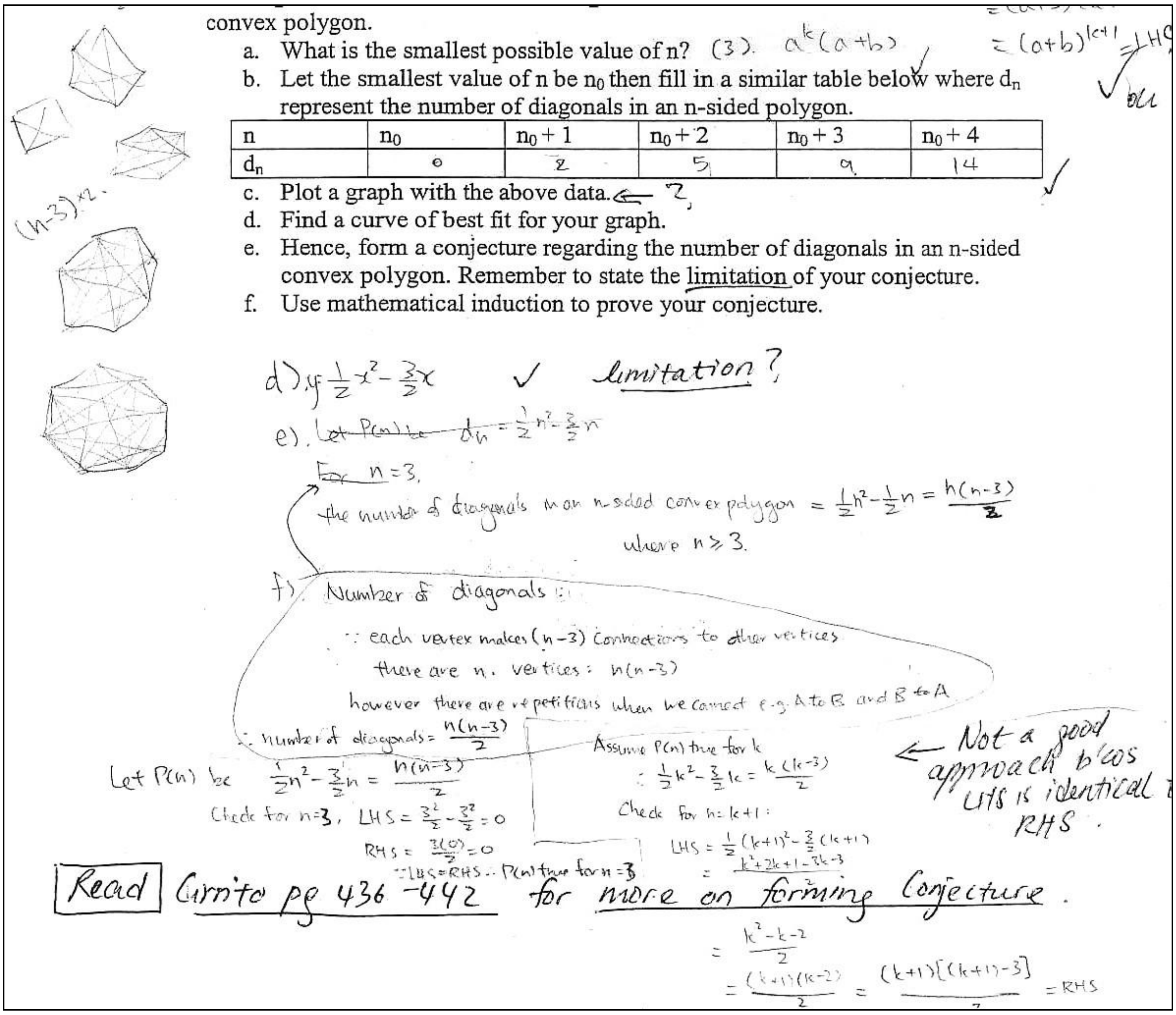
∴ d(k+1) is true ✓

By M.I. d(n) is true for all $n \geq 4$

Excellent work!

Exhibit 2

Instead of using recurrence equations, some students used counting principles in their proofs. Here are two examples and both, in effect, are constructive proofs. Three students worked in a Group C and argued that a diagonal is formed when a vertex is linked to another vertex that is not its immediate neighbours (see exhibit 3). Two vertices are immediate neighbours if the edge linking them forms part of the perimeter of the convex *n*-sided polygon. Given n, the number of vertices, each vertex forms $(n-3)$ edges with other vertices. As each diagonal is counted twice, the number of diagonals is simply $\frac{n(n-3)}{2}$. Since they tried a different approach without using the recurrence equation $d_n = d_{n-1} + (n-2)$, they were not able to prove the result by mathematical induction.



convex polygon.

a. What is the smallest possible value of n? (3).

b. Let the smallest value of n be n_0 then fill in a similar table below where d_n represent the number of diagonals in an n-sided polygon.

n	n_0	$n_0 + 1$	$n_0 + 2$	$n_0 + 3$	$n_0 + 4$
d_n	0	2	5	9	14

c. Plot a graph with the above data.

d. Find a curve of best fit for your graph.

e. Hence, form a conjecture regarding the number of diagonals in an n-sided convex polygon. Remember to state the limitation of your conjecture.

f. Use mathematical induction to prove your conjecture.

d) $y = \frac{1}{2}x^2 - \frac{3}{2}x$ ✓ limitation?

e) For n = 3, the number of diagonals in an n-sided convex polygon $= \frac{1}{2}n^2 - \frac{3}{2}n = \frac{n(n-3)}{2}$ where $n \geq 3$.

f) Number of diagonals: ∵ each vertex makes (n − 3) connections to other vertices, there are n vertices: n(n − 3); however there are repetitions when we connect e.g. A to B and B to A. ∴ number of diagonals $= \frac{n(n-3)}{2}$

Let P(n) be $\frac{1}{2}n^2 - \frac{3}{2}n = \frac{n(n-3)}{2}$

Check for n = 3, LHS $= \frac{3^2}{2} - \frac{3^2}{2} = 0$, RHS $= \frac{3(0)}{2} = 0$ ∴ LHS = RHS ∴ P(n) true for n = 3

Assume P(n) true for k: $\frac{1}{2}k^2 - \frac{3}{2}k = \frac{k(k-3)}{2}$

Check for n = k + 1: LHS $= \frac{1}{2}(k+1)^2 - \frac{3}{2}(k+1) = \frac{k^2+2k+1-3k-3}{2} = \frac{k^2-k-2}{2} = \frac{(k+1)(k-2)}{2} = \frac{(k+1)[(k+1)-3]}{2}$ = RHS

← Not a good approach b'cos LHS is identical to RHS.

Read Grrrito pg 436 – 442 for more on forming Conjecture.

Exhibit 3

Student D also used counting principles but he took a different approach (see exhibit 4). He argued that the number of edges that could be drawn with any two vertices out of n possible vertices is ${}^{n}C_2$. Since the perimeter of an *n*-sided polygon is n, then the number of diagonals is simply

$$d_n \quad = \quad {}^{n}C_2 - n \; ; \; n \geq 3 \quad \text{..(3)}$$

Similar to the previous case, Student D was unable to prove the result by mathematical induction. Instead he tried to prove ${}^{n}C_2 - n = \frac{n(n-3)}{2}$ by mathematical induction.

Having said that, both attempts were praiseworthy because they used their knowledge from counting principles that were covered in previous lessons to synthesize a creative solution to the given task. In fact, a direct proof in which a result is derived directly from logic or known premises is preferred to proofs using mathematical induction. Therefore, both attempts by Group C and Student D were of higher order and represented adequate proofs of the result even without the part with mathematical induction.

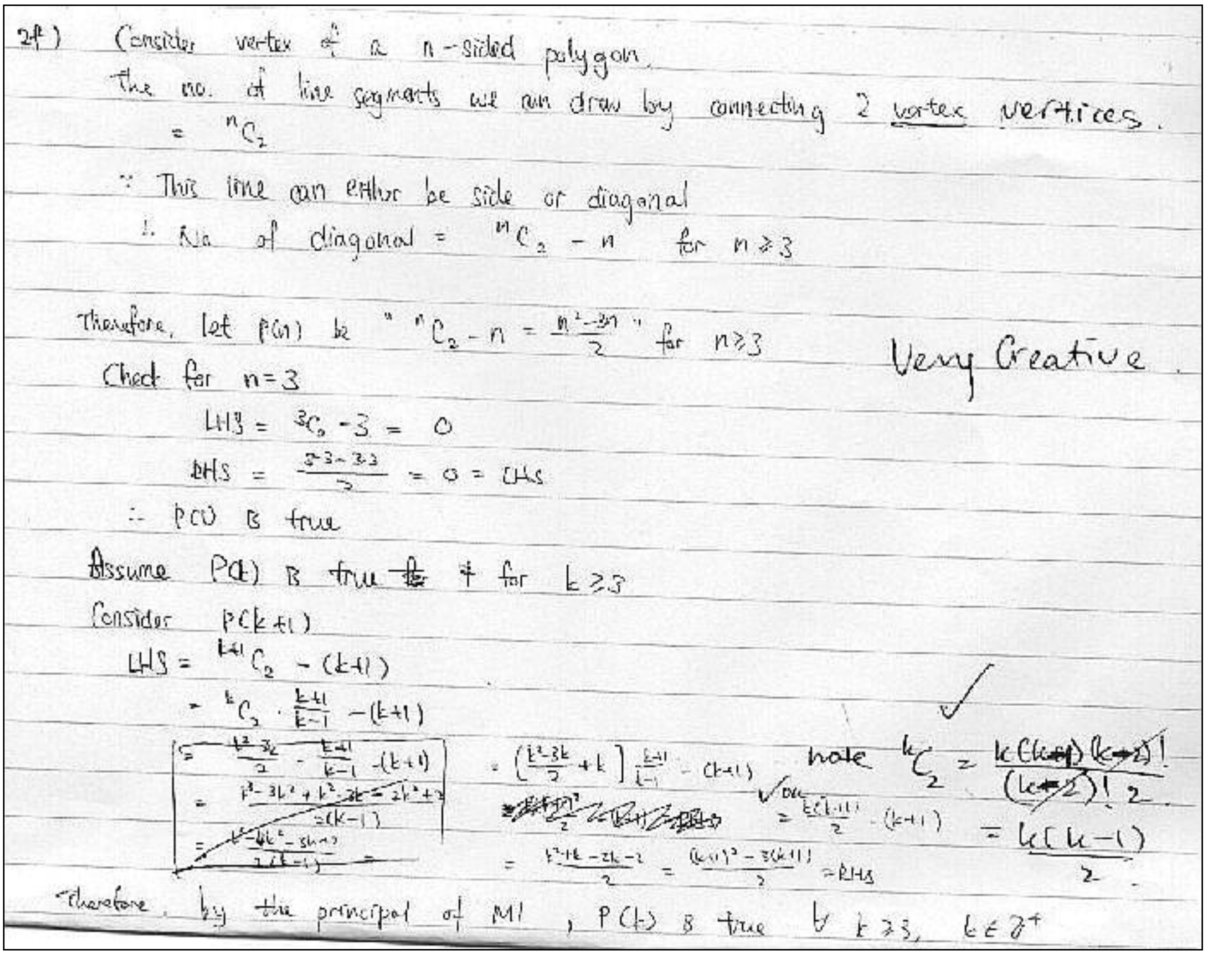

2f) Consider vertex of a n-sided polygon.
The no. of line segments we can draw by connecting 2 vertex vertices.
$= {}^nC_2$
∵ This line can either be side or diagonal
∴ No. of diagonal $= {}^nC_2 - n$ for $n \geq 3$
Therefore, let P(n) be " ${}^nC_2 - n = \frac{n^2-3n}{2}$ " for $n \geq 3$
Very Creative.
Check for n = 3
LHS $= {}^3C_2 - 3 = 0$
RHS $= \frac{3\cdot 3 - 3\cdot 3}{2} = 0 =$ LHS
∴ P(3) is true
Assume P(k) is true for $k \geq 3$
Consider P(k+1)
LHS $= {}^{k+1}C_2 - (k+1)$
$= {}^kC_2 \cdot \frac{k+1}{k-1} - (k+1)$
$= \left(\frac{k^2-3k}{2} + k\right)\frac{k+1}{k-1} - (k+1)$
$= \frac{k(k+1)}{2} - (k+1)$
$= \frac{k^2+k-2k-2}{2} = \frac{(k+1)^2 - 3(k+1)}{2} =$ RHS
note ${}^kC_2 = \frac{k(k-1)(k-2)!}{(k-2)!\,2} = \frac{k(k-1)}{2}$
Therefore, by the principal of MI, P(k) is true ∀ $k \geq 3$, $k \in Z^+$

Exhibit 4

<u>3. Other Methods of Forming Conjecture</u>

Equation (2) above can also be obtained in a fashion consistent with the tradition of graph theory. Let us assume that d_n is the number of diagonals in an n-sided convex polygon. We will now remove a vertex from this n-sided convex polygon and the polygon is reduced to $(n - 1)$ edges. Out of these $(n - 1)$ edges, two of these used to make up the perimeter of the previous n-sided convex polygon. Hence, the removal of a vertex only reduces $(n - 3)$ diagonals. At the same time, one of the previous diagonals from the n-sided convex polygon now forms part of the perimeter of the new (n – 1)-sided convex polygon. Hence, the process of removing a vertex from an n-sided convex polygon, in effect, reduces the number of diagonals by (n – 3 + 1) or (n – 2) . We thus obtain equation (2).

Cirrito (2004, p. 437) uses the recurrence equation

$$d_{n+1} \quad = \quad d_n + (n - 1)\ ;\ \ n \geq 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Instead of removing a vertex, we can consider adding an additional vertex to an n-sided convex polygon. This process creates an addition of n edges with a new edge for each of the n vertices. Out of these new n edges, two are linked to immediate neighbours to form part of the perimeter of the new (n + 1)-sided convex polygon. Furthermore, one of the edges that form part of the perimeter of the previous n-sided convex polygon is now a diagonal. At the end, the number of new diagonals formed by this additional vertex is (n + 1 – 2) or just (n – 1) . Hence, we obtain equation (4) above.

The conjecture represented by equation (1) can also be obtained by manipulating arithmetic series. We will first tabulate the results for the number of diagonals d_n for various n-sided convex polygons as in Table 1.

n	3	4	5	6	7
d_n	0	2	5	9	14
T_n		2	3	4	5

Table 1

We form a new sequence such that $T_n = d_n - d_{n-1}$ as in Table 1 above. This new sequence T_n is an arithmetic sequence with common difference 1 . Thus, $T_n = 2 + (n - 4)\ ;\ n \geq 4$. It then follows that:

$$d_4 \quad = \quad d_3 + T_4$$

$$d_5 \quad = \quad d_3 + T_4 + T_5$$

$$d_6 \quad = \quad d_3 + T_4 + T_5 + T_6$$

Since $d_3 = 0$, we conjecture that

$$\begin{aligned} d_n &= \sum_{i=4}^{n} T_i \ ; \ n \geq 4 \\ &= \frac{(n-4+1)}{2}(2+2+n-4) \ ; \ n \geq 4 \\ &= \frac{n(n-3)}{2} \ ; \ n \geq 4 \end{aligned}$$

Thus, the number of diagonals of an *n*-sided convex polygon is

$$d_n = \frac{n(n-3)}{2} \ ; \ n \geq 3$$

Similar to the method above, we could also use the Method of Finite Differences in forming our conjecture. From Table 1 above, we need to use two rows of differences between two consecutive terms before we obtained a row where all the differences are equal to a constant, in this case, 1. Thus, d_n in this case must be of the form $an^2 + bn + c$ and the remaining task is to determine coefficients a, b and c. Using the information from Table 1 above, we can obtain the following system of equations:

$$\begin{aligned} 0 &= 9a + 3b + c \\ 2 &= 16a + 4b + c \\ 5 &= 25a + 5b + c \end{aligned}$$

Solving these simultaneous equations, we obtain $a = \frac{1}{2}$, $b = -\frac{3}{2}$ and $c = 0$. Hence, we obtain equation (1).

4. Conclusion

The above assignment provides an opportunity for students to see Mathematics not only in terms of calculation but also as a body of interlocking ideas. Students who are already proficient with the manual steps of mathematical induction can be encouraged to employ this technique in a creative manner and explore mathematical ideas. Finally, it is the intention of this note to highlight some of the methods that can be used to form a conjecture for the number of diagonals in an *n*-sided convex polygon.

Appendix 1

HL: Assignment on Mathematical Induction

1. Use Mathematical Induction to prove that $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$.

2. Investigation: Find the number of diagonals that can be drawn in an n-sided convex polygon.

   (a) What is the smallest possible value of n ?

   (b) Let the smallest value of n be n_0 then fill in a similar table below where d_n represents the number of diagonals in an n-sided polygon.

   (c) Plot a graph with the above data.

   (d) Find a curve of best fit for your graph.

   (e) Hence, form a conjecture regarding the number of diagonals in an n-sided convex polygon. Remember to state the limitation of your conjecture.

   (f) Use mathematical induction to prove your conjecture.

Read Cirrito pp 436 – 442 for more on Forming Conjecture.

Reference

Cirrito, F. (Ed.) (2004). *Mathematics Higher Level (Core)*. Victoria: IBID Press.

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