Mining the Quadratic Formula: An Integrated Approach to Quadratic Function

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1. Introduction

Mathematic lessons should not be presented to students as introduction of concepts or methods follows by hours of senseless calculations. Mathematics is about ideas and how different ideas relate to each other (Stewart, 1996). Students probably can experience a meaningful lesson if they can see mathematics as a game of logic and connections. This little note attempts to introduce an example of an integrated approach to quadratic function through "mining" the quadratic formula. The word "mining" suggests an attempt to obtain as much information as possible from the quadratic formula and creative ways may be employed.

This approach is probably more appropriate for students who already have some exposure to quadratic function and are familiar with the quadratic formula. I used this approach with my year one students in Mathematics Higher Level at the International Baccalaureate Diploma level. But I believe this approach will also benefit students at Form 5 in the local school system as a revision tool.

The lesson aims to link different concepts related to quadratic functions by focusing on the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1}$$

The lesson highlights the important features of the quadratic function. Namely,

(a) the root or roots (point A and point B in figure 1),

(b) the axis of symmetry (the dotted line in figure 1), and

(c) the turning point or vertex (point D in figure 1).

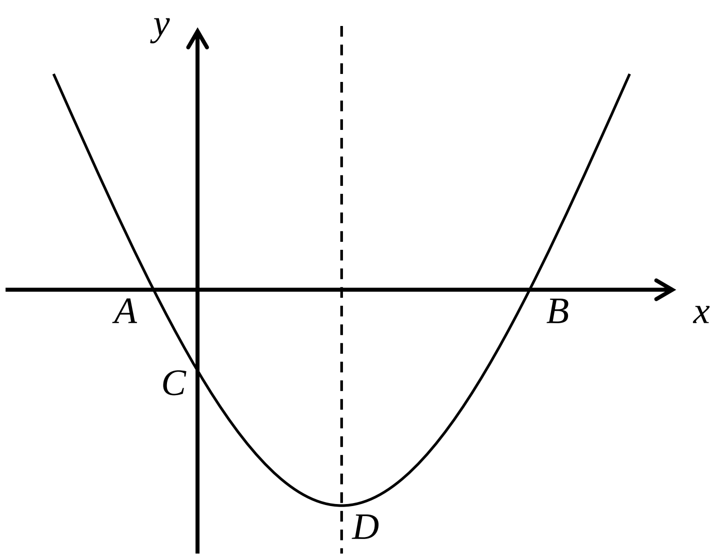

Figure 1

This integrated approach will also attempt to link:

(a) the method of completing the square with the quadratic formula, and

(b) the concept of discriminant with the quadratic formula.

2. The origin of the quadratic formula

The first place to start is probably to ask the question of origin, "Where does the quadratic formula come from?" The question is to help instill in students the habit of asking for a proof to any given mathematical formula. After all, if this formula is not valid in the first place then there is not much point "mining" it. Most teacher of Mathematics will immediately recognize that one effective strategy here involves completing the square for $ax^2 + bx + c = 0$. However, the teacher should not provide the solution but encourage students to consider the problem. My students gave it some thoughts and after a few minutes, one student finally provided the proof using completing the square. From my experience, students may also take a different route by assuming the formula is true and see whether or not they can rearrange the formula back to $ax^2 + bx + c = 0$. With this exercise, we can also question students for their meaning of "solve." According to Collins Dictionary of Mathematics, one of the meanings of "solve" is "to find the value or sets of values of the variables that **satisfy** (an equation or system of equations)." Obviously, in this case, "solve" carries the implicit command of equating the quadratic formula with zero. Thus, this first exercise involves the habit of asking for evidence for a claim, the meaning of "solve" and the ability to apply the method of completing the square.

3. Roots, discriminant and the quadratic formula

Meaningful learning is attained when a student can connect new information with his or her existing schema (Slavin, 2003). An effective Mathematic teacher should help students make sense of terminology like root and discriminant. Root in English has the meaning of origin. According to The Random House College Dictionary, root is the number that one multiplies repeatedly to obtain other number. This is also called the radical that is derived from *radix* in Latin and *radix* means root. Mohammed ibn Musa al-Khwarizmi in Algebra also used root to indicate a value that will produce zero for a particular function (Ballew, 2004). Following al-Khwarizmi, one can consider the x-axis as the ground level and the x-intercept as a point embedded in the ground, thus root. By this latter definition, the x in the quadratic formula is a root.

The value $b^2 - 4ac$ in the quadratic formula is known as the discriminant, and is denoted by Δ. A teacher may find it useful to associate the term discriminant with the English word "discriminate" which means to make a clear distinction or to judge wisely. Thus, the discriminant **determines** or **helps us to judge** the nature of our root. We can rewrite the quadratic formula into equation (2) to highlight the importance of the discriminant.

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} \tag{2}$$

It is perhaps clear from (2) that we can list out the following properties:

$\Delta < 0$	Two distinct complex roots. No real root.
$\Delta = 0$	The roots are identical and $x = -\dfrac{b}{2a}$.
$\Delta > 0$	There are two real roots.

4. The line of symmetry and the quadratic function

The line of symmetry is probably a bit neglected in the study of quadratic function at higher secondary school level. Figure 1 shows that this line is at equal distance from the two distinct roots. If the quadratic function does not have two real roots then we can still identify the line because it passes through the vertex. In fact, the line of symmetry can be expressed as x = (x-coordinate of the vertex).

Applying the method of completing the square to $ax^2 + bx + c = 0$,

we obtain
$$a\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a} = 0$$
$$a\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a}$$

Hence, the vertex is $\left(-\frac{b}{2a}, \frac{-(b^2-4ac)}{4a}\right)$ or $\left(-\frac{b}{2a}, \frac{-\Delta}{4a}\right)$.

However, this vertex can be obtained from formula (2) with minimum calculation. Observe that the x-coordinate and the line of symmetry is just the part in (2) enclosed by the "sock" in figure 2. For students who have difficult remembering that the sock is the x-coordinate of the vertex, the teacher can emphasize that this is actually a Santa's Sock for "X-mas." The "X" in "X-mas" relates the sock to the x-coordinate of the vertex. To summarize, $x = -\frac{b}{2a}$ is the line of symmetry, the x-coordinate of the vertex, and the x-coordinate when the discriminant is zero (section 3 above).

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

Figure 2

The y-coordinate of the vertex comes from the part of the numerator left out from the sock, but in this case, $y = -\left(\frac{\Delta}{2a}\right)\frac{1}{2}$. A teacher can help students to memorize this y-coordinate by sharing this following nonsense.

"The y-coordinate is not simply $\left(\frac{\Delta}{2a}\right)$ because there is only one sock in the formula, thus it has to be multiplied by **half** of a pair of socks, thus, $\frac{1}{2}$. But you also need to add a negative sign here because **half** of a sock is not as positive or good as having a pair of socks." A teacher can further point out another observation. The sock in the left is the x-coordinate whereas the "un-socked" part on the left is the y-coordinate and this order replicates the pattern in an order pair (x, y) .

Is there other way to make sense of the sock, $x = -\frac{b}{2a}$? Sure, assuming that there are two distinct roots in a quadratic function, the line of symmetry is located at equal distance from these roots. Thus, we can apply the concept of midpoint to obtain our desired outcome.

Let x_1 and x_2 be two different roots of this quadratic function. We obtain $x_{\text{axis}} = \frac{x_1 + x_2}{2}$.

Using the quadratic formula, we can set $x_1 = \frac{-b-\sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b+\sqrt{\Delta}}{2a}$.

Hence, $x_{\text{axis}} = \frac{-b-\sqrt{\Delta}-b+\sqrt{\Delta}}{4a} = -\frac{b}{2a}$.

Since the line of symmetry passes through the vertex then $x = -\frac{b}{2a}$ can also be obtained by taking the derivative of $y = ax^2 + bx + c$ with respect to x , setting the derivative to zero, and then solving for x . Retracing this process, we obtain $\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax - b$ which is a creative rearrangement of the sock. Furthermore, the second derivative of the quadratic function $y = ax^2 + bx + c$ is simply the denominator of the sock. Another trivial of interest is that the sock is half of the sum of the two roots in a quadratic function. That is, if α and β are two roots of a quadratic function then $\frac{1}{2}(\alpha + \beta) = -\frac{b}{2a}$. Teachers may be surprised to find that most students do not make these connections.

5. Conclusion

This note is an attempt to highlight different ideas that can be connected using the quadratic formula as our starting point. One can also consider this as an exercise of getting as much information as possible out from the quadratic formula.

References

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