EduMath 20 (6/2005)

Introduction to Conics with Cabri 3D

In memoriam Miguel de Guzmán (1936 – 2004) who significantly enriched the didactics of mathematics.

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1. Introduction

The history of conics in the mathematics classroom is a history of ups and downs (see, e.g. Schupp 1988). The most recent instance is another threat of neglect as the focus of mathematics teaching is being shifted from mathematical substance to other issues (e.g. mathematics literacy – PISA) – despite the fact that conics are an excellent example of the interdependence between geometry and algebra. Other arguments in favour of conics are their practical relevance (satellite orbits, shot-putting, lithotriptor, ...), internal mathematical relevance, richness of problems, variety of methods etc.

New media, in particular dynamic geometry systems, offer new ways of treating conics, which aids the teaching and learning of that topic. While the well-researched didactic monography on conics by Schupp (1988) still focused on the programming of the graphic visualisation, Schumann (1991) was the first to point out the interesting applications of dynamic geometry for the treatment of conics. So far, most contributions on conics have been limited to a two-dimensional treatment, which is a paradox in view of the nature of the subject and indeed of its very name. This neglect may be due to the deficiencies of conventional methods, which provide no appropriate learning environment for three-dimensional treatment of conics. Cabri 3D is the first efficient dynamic spatial geometry system that fits this purpose. The following is to provide an adequate introduction to spatial conics using Cabri 3D.

2. Introducing conics in space

For an adequate dynamic introduction of conics as two-dimensional sections of a (double) cone, we need a 3-dimensional dynamic geometry system which offers the user good perceptibility and/or visualisation of three-dimensional objects, including the options of construction, direct manipulation and variation. Cabri Géomètre 3D (Laborde, J. M.; Bainville, E., Cabri Géomètre 3D, 2004) is the first three-dimensional dynamic geometry system which even in version 1.0 meets many of the requirements on a tool for three-dimensional synthetic geometry for dynamic treatment, similar to the options offered by 2D dynamic geometry systems.

Cabri 3D provides better insight into two-dimensional conic sections and the foundation of the phenotypical shapes of the curve of intersection than traditional media or different new media.

The following illustrations provide only a limited idea of the dynamic options of Cabri 3D and the resulting visual perceptions and experiences.

First let us, in Cabri 3D, construct a cone from a circle in a plane and a point in the line normal to the plane constructed in the centre of the circle. (Figure 1, model of a circular cone; all the figures in this article are originally in informative colours). Its shape can be varied by moving this point (the apex of the cone – Figure 2, modified cone shape).

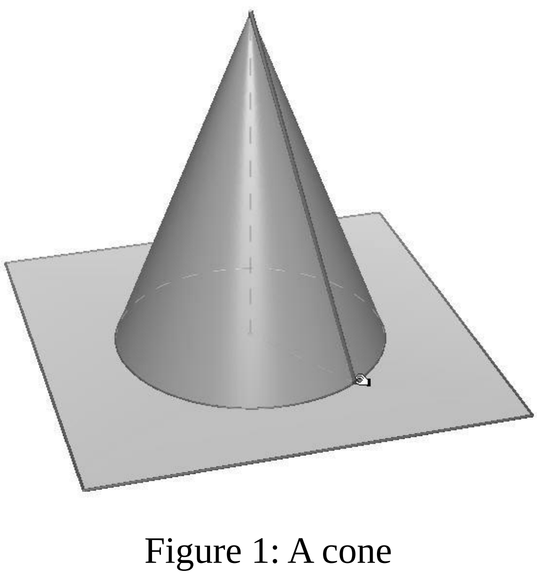

Figure 1: A cone

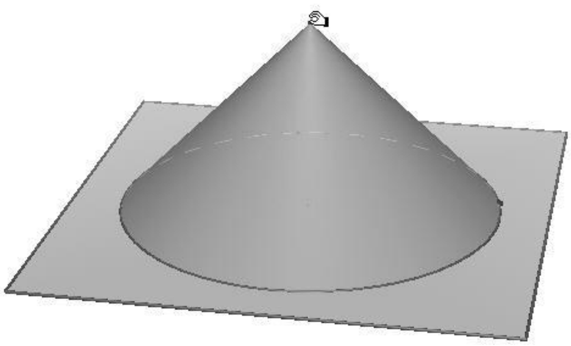

Figure 2: Varying a cone

We can now look at the cone from all sides using the so-called virtual sphere device implemented in Cabri 3D.

Now, we are going to construct a plane in the cone and let the system generate the curve of intersection, which in our case Cabri 3D identifies as an ellipse (Figure 3), irrelevant of the shape of the cone. We can also look at the curve of intersection from below (Figure 4).

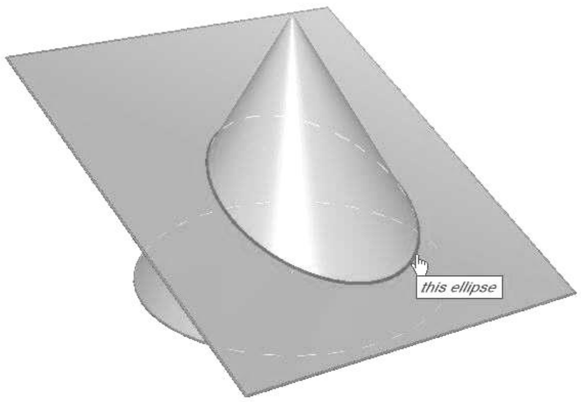

Figure 3:
A plane section generating an ellipse

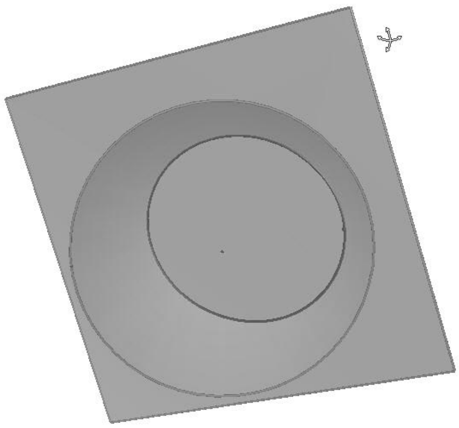

Figure 4: Elliptic section from below

We are now going to rotate the image so that the plane is viewed as a straight line. We find that the base plane has a smaller angle with the intersecting plane than with the lateral surface (Figure 5).

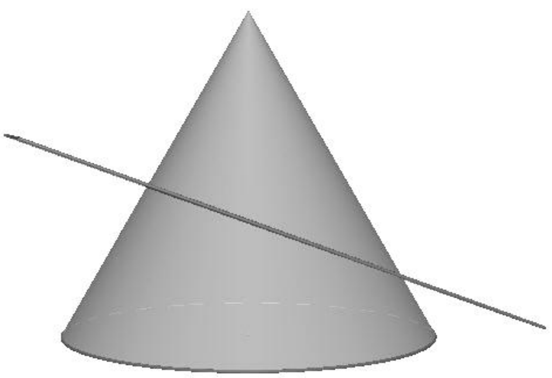

Figure 5: Position of plane trace for an elliptic section

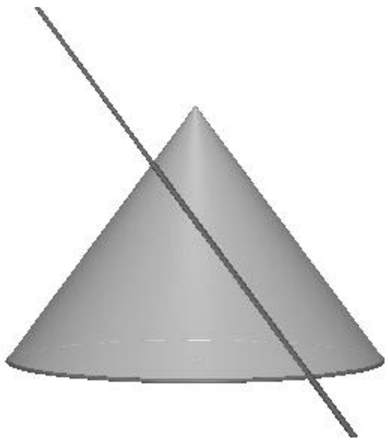

Figure 6: Position of plane trace for a parabolic section

For a parabolic section, we construct a plane exactly parallel to a generating line of the cone; figure 6 shows the trace of this plane.

Cabri 3D identifies the curve of intersection generated with this plane as a parabola (Figure 7).

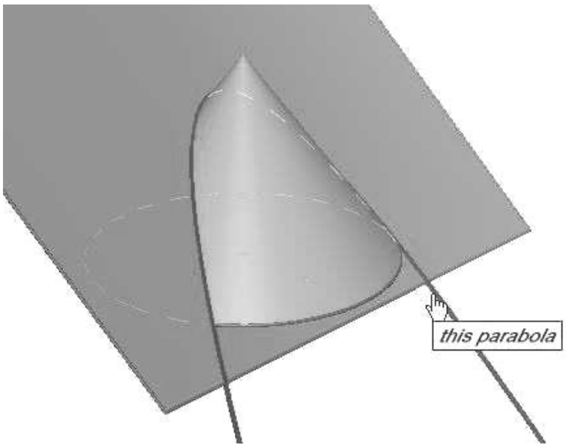

Figure 7: A parabola by plane section

By varying the position of the intersecting plane so that its angle with the base plane is larger than the angle of the lateral surface, we get a hyperbola. The second branch of the hyperbola is located on a cone generated by reflecting in the apex of the original cone (Figure 8: trace of the intersecting plane in the double cone; Figure 9: hyperbola as curve of intersection in the double cone). Again, we can look at the image from all sides (Figure 10, e.g. from below).

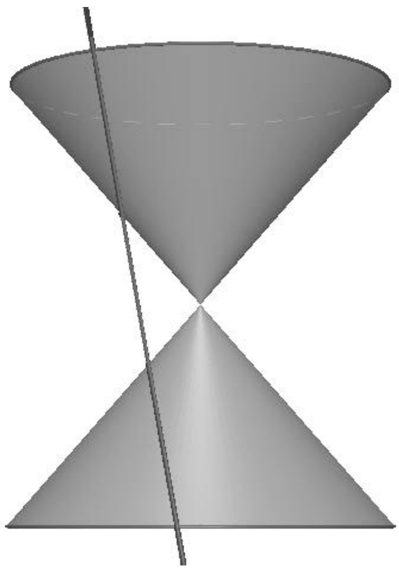

Figure 8: Position of plane trace for a hyperbolic section

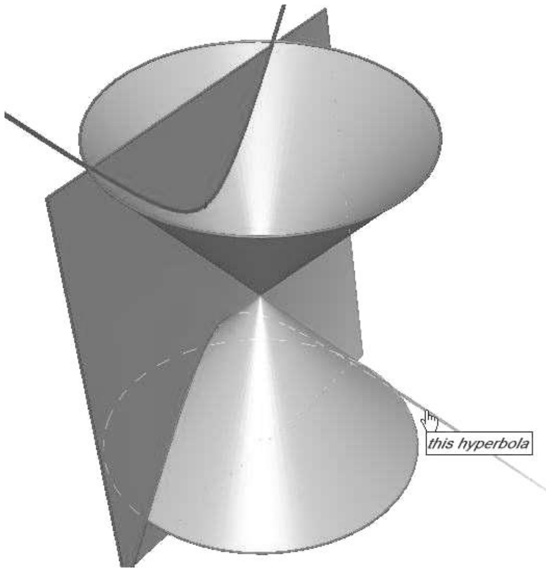

Figure 9:
A hyperbola by plane section

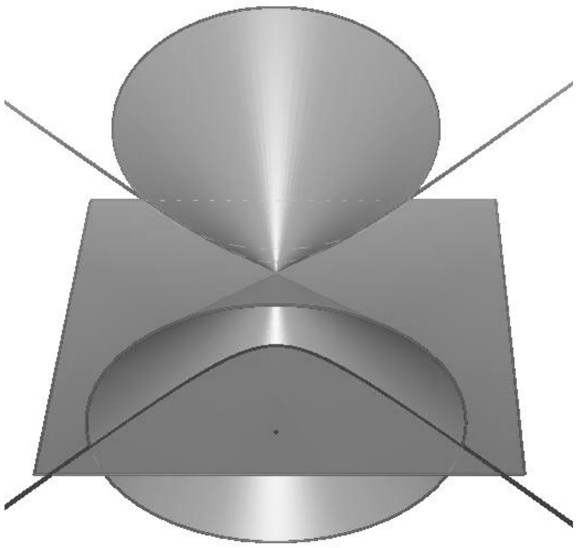

Figure 10: A hyperbola by plane section from below

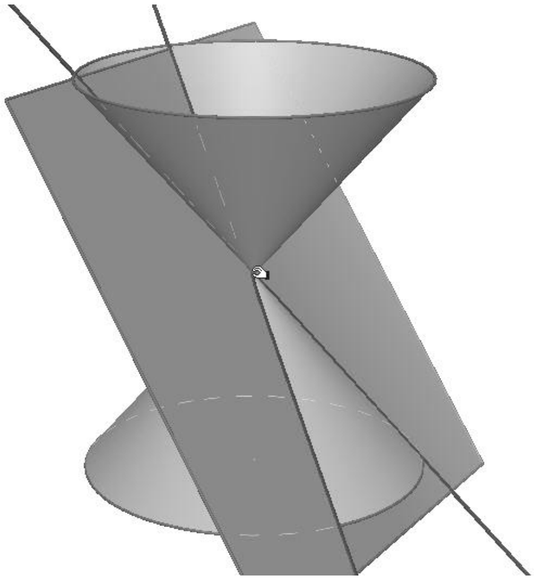

Figure 11: Borderline case

If the intersecting plane passes through the apex, we get a double straight line, (Figure 11) etc.

We are now going to show why, e.g. the curve of intersection of Figure 12 meets the point curve characteristics of an ellipse, i.e. that the sum of the distance from two fixed points is constant for all points of the curve. For this, we are going to use the elegant method first described by Dandelin 1794 – 1847 (see, e.g. Manual of School Mathematics, Vol. 4, p. 116/117).

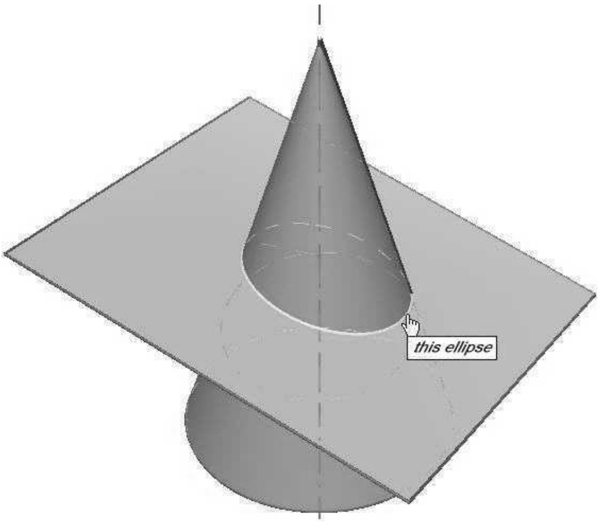

Figure 12: Elliptic section?

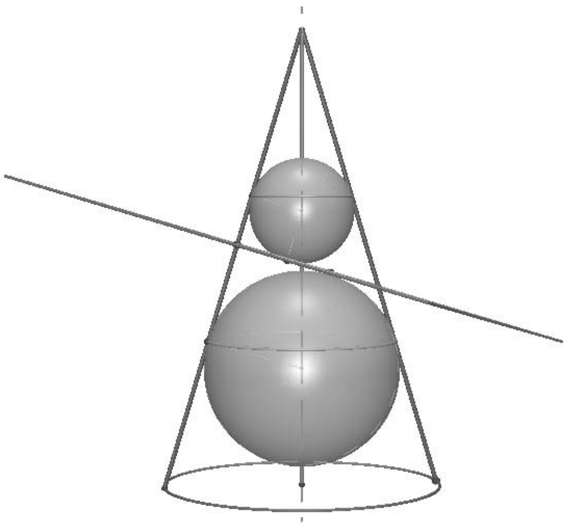

Figure 13: A cone with intersecting plane trace tangent to inscribed spheres

First, we are going to construct the so-called Dandelin spheres, i.e. the spheres inscribed in the cone which are tangent to the intersecting plane. For better visualisation, we will show only the outlines of the cone (Figure 13, construction of the cones with Dandelin spheres, tangent points, trace of intersecting plane and the inscribed circles; construction lines are omitted for better visualisation; figure 14 from a different view; figure 15 Dandelin spheres from below).

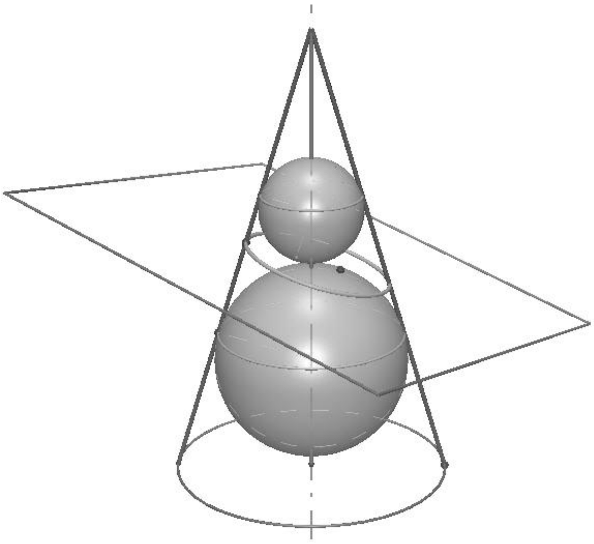

Figure 14: Figure 13 from beyond

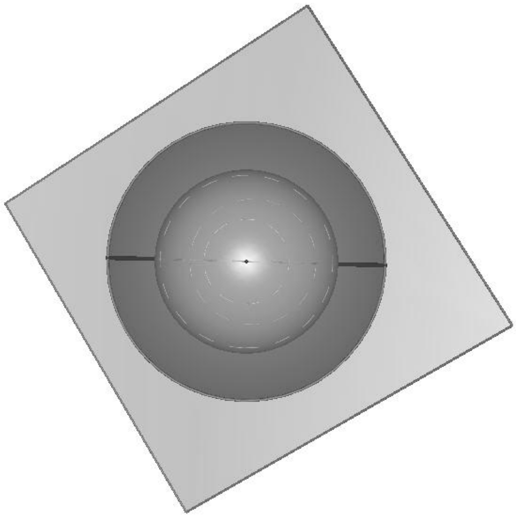

Figure 15: Figure 13 from below

The centres of a sphere are constructed in a plane containing the cone axis in analogy to the construction of a circle centre from two intersecting tangents and a diameter. The radii of the spheres are obtained by raising perpendiculars from the centres of the spheres to a generating line of the cone. The points where the perpendiculars pass through the intersecting plane are the tangent points (later, the foci of the ellipse). On the curve of intersection, we put a moving point P .

For our proof, we shall name some further objects, and we are going to show only the outlines of the spheres for better visualisation. (Figure 16, enlarged detail in figure 17): F_1 , F_2 are the tangent points; k_1 , k_2 the tangent circles; B_1 , B_2 the points of intersection of the generating line going through P with the tangent circles k_1 , k_2 ; A_1 , A_2 are the end points of a generating line of the truncated cone formed by k_1 , k_2 .

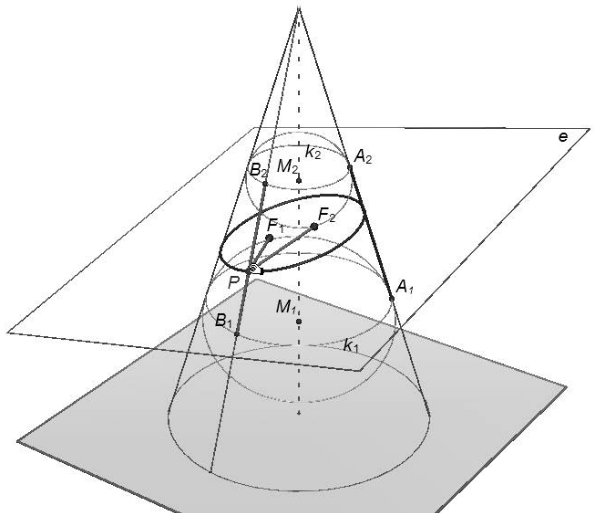

Figure 16:
Proofing figure with denotations

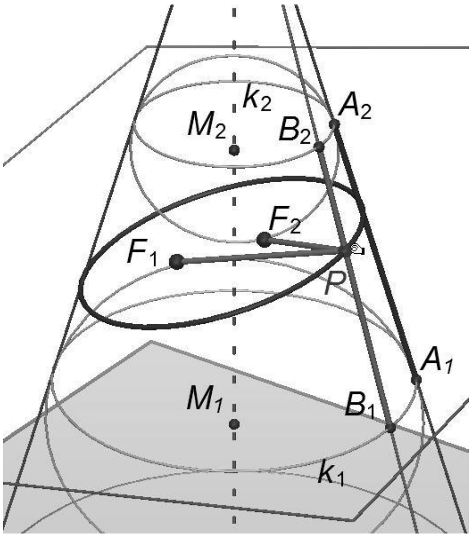

Figure 17:
Proofing figure zoomed

If we move P, B_1B_2 along the lateral surface of this truncated cone; B_1B_2 becomes congruent with A_1A_2.

B_1, F_1, respectively B_2, F_2, are tangent points with the first or second Dandelin sphere; together with P they form tangential sections located on the tangent cone with P as the apex (e.g. Figure 18 for the first sphere; construction of the cone with the three-point circle through B_1, F_1 and F_1'; F_1' is obtained by reflecting F_1 on the plane through P and the cone axis).

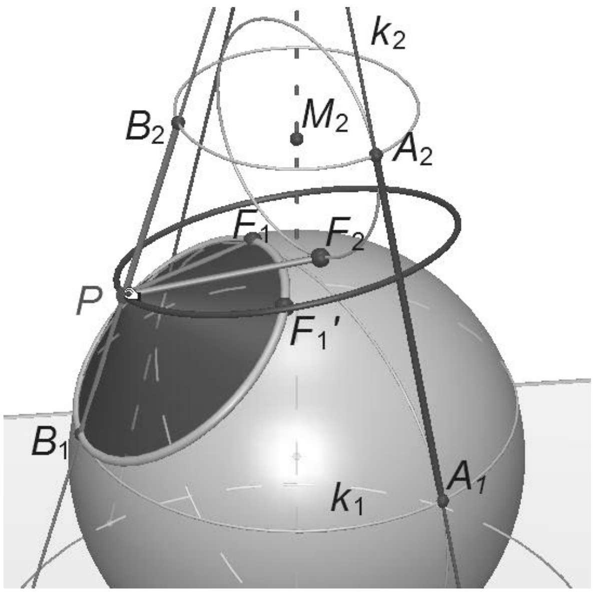

Figure 18: The tangent cone

PB_1 and PF_1 are thus of equal length, as are PB_2 and PF_2. This means that

$$PF_1 + PF_2 = PB_1 + PB_2 = A_1A_2 = \text{constant}$$

and the curve of intersection is an ellipse. We find that the generating line of the truncated cone formed by the tangent circles k_1, k_2 are identical in length with the main axis of the ellipse to be drawn.

The proofs of the parabola and hyperbola sections are obtained by a similar procedure.

3. Another view: Conics as circle images in central projection

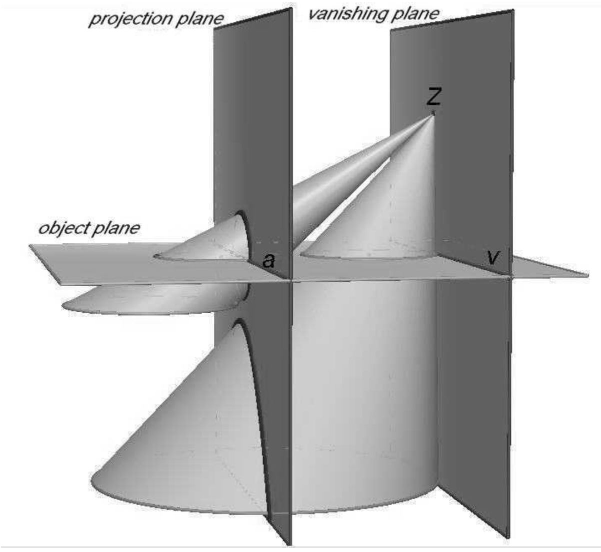

Figure 19: Ellipse and parabola as central-projective images of a circle

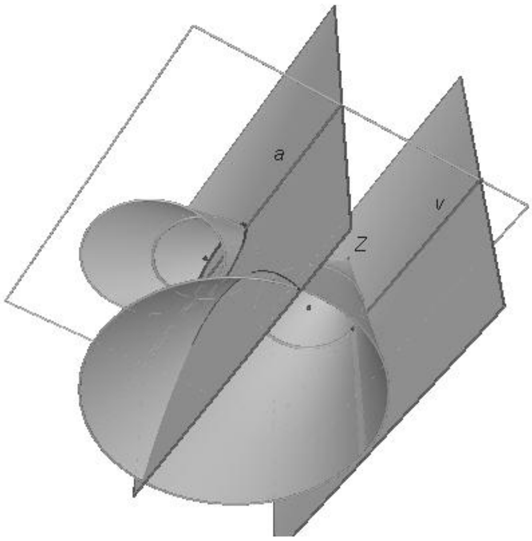

Figure 20: Figures 19 from below

Naturally, Cabri 3D is suited also for 3D construction and visualization of conics as circle images in central projection in the common manner of representation, i.e. with a central point Z (eye point respectively point of sight), the object plane of the projection plane (which intersects with the object plane in the axis a), and the vanishing plane (with the vanishing line v). Figure 19 shows how the cone constructed from a circle in the object plane with the apex Z intersects with the projection plane in an ellipse. For a circle tangent to the vanishing plane, the curve of intersection will be a parabola. The image can be viewed from all sides (Figure 20, from “below”, with a transparent object plane). Figures 21 and 22 show the image obtained for a hyperbola section.

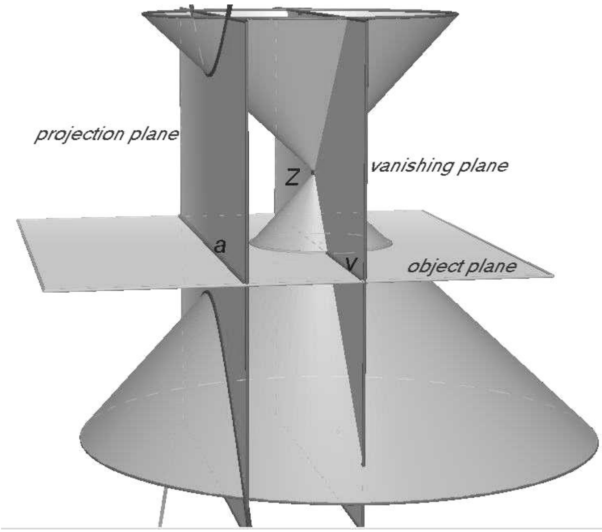

Figure 21: Hyperbola as central-projective image of a circle

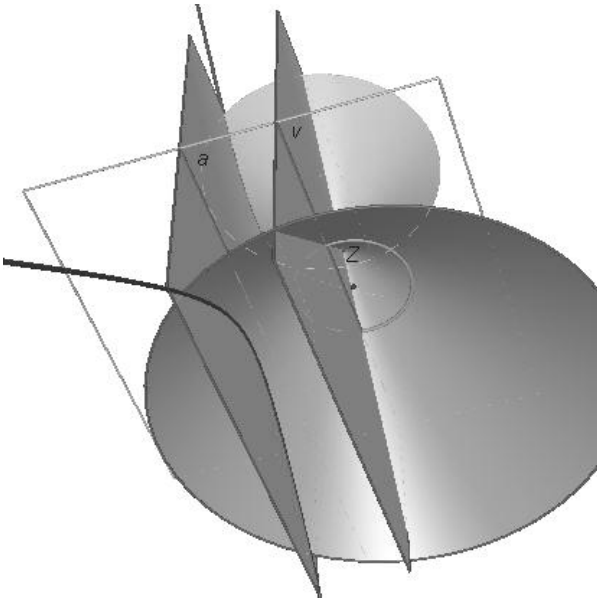

Figure 22: Figure 21 from below

Figure 23 is the configuration obtained for point-by-point construction of a circle image in central projection, for the example of an ellipse (*k*: original, *k*': image; the bottom point of the perpendicular from *Z* onto projection plane is the main point *H*, and the plane parallel to the object plane going through *Z* intersects with projection plane in the horizon *h*). To construct the image *k*' of the circle *k* , we drop the perpendicular from the point *P* on the circle onto the axis *a* , with *A* as the bottom point of the perpendicular, ... ; *ZHAV* is a rectangle after construction. The point of intersection of the projection ray *ZP* with the straight line *AH* is the image point *P*' which generates the circle image as *P* moves along the circle. The projection ray *ZP* is a generating line describing the oblique circular cone formed by *Z* and *k* . After masking the image plane and cone, the scenario of figure 24 is obtained.

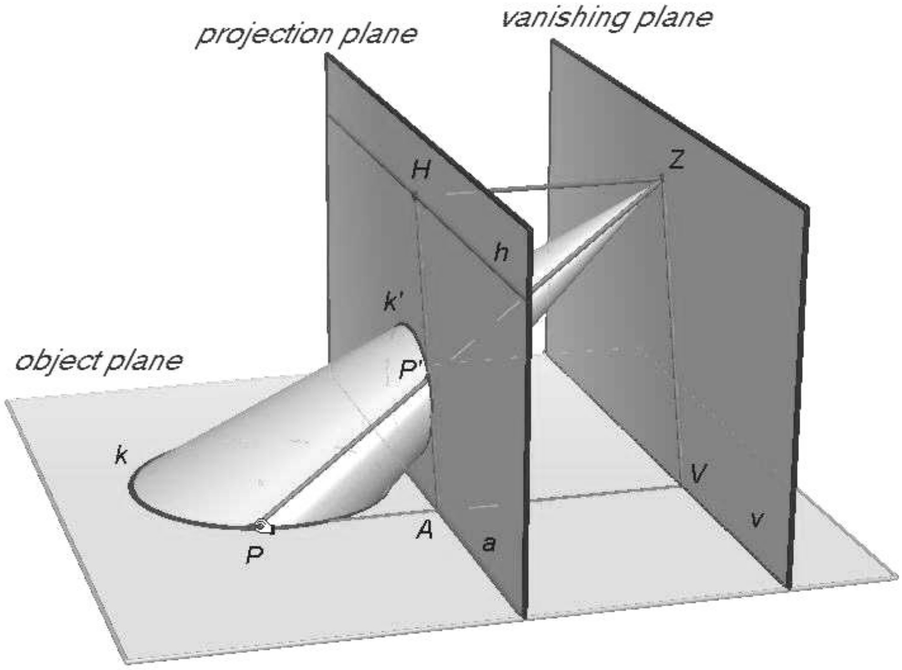

Figure 23: Ellipse as central-projective image of a circle with corresponding denotations

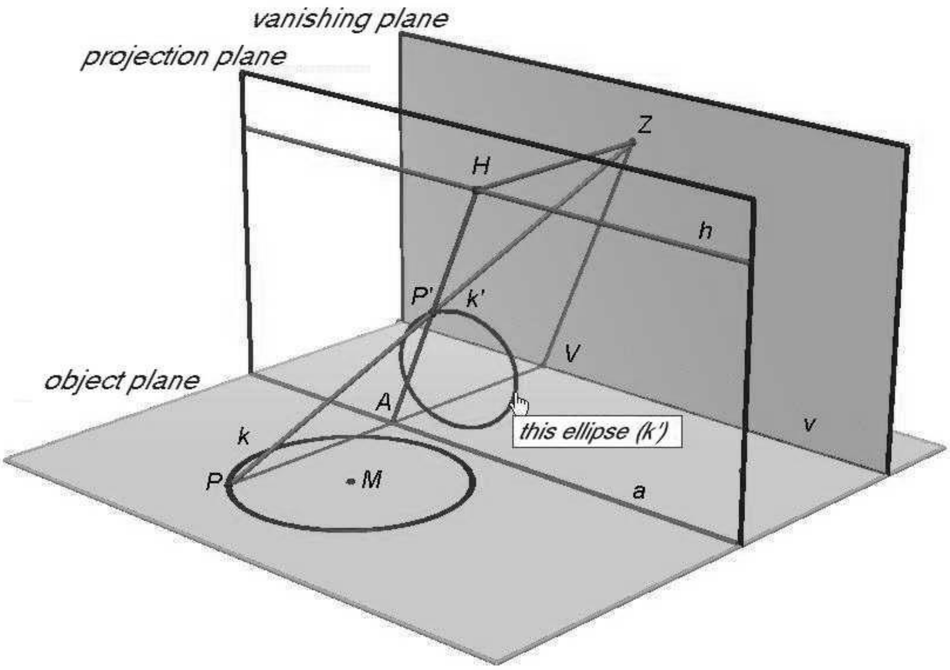

Figure 24: Figure 23 with transparent projection plane

By tilting both the projection plane and vanishing plane into the object plane (Figure 25), we obtain a 2-dimensional construction of circle images in central projection as a special case of an axial projective transformation (Figure

26). We are now going to use Cabri II Plus to apply this plane image, with distinctions made for different cases, to circles for generating conics. Circle does not intersect or contact the vanishing line (ellipse: Figure 27), contacts it (parabola: Figure 28) or intersecting it (hyperbola: Figure 29).

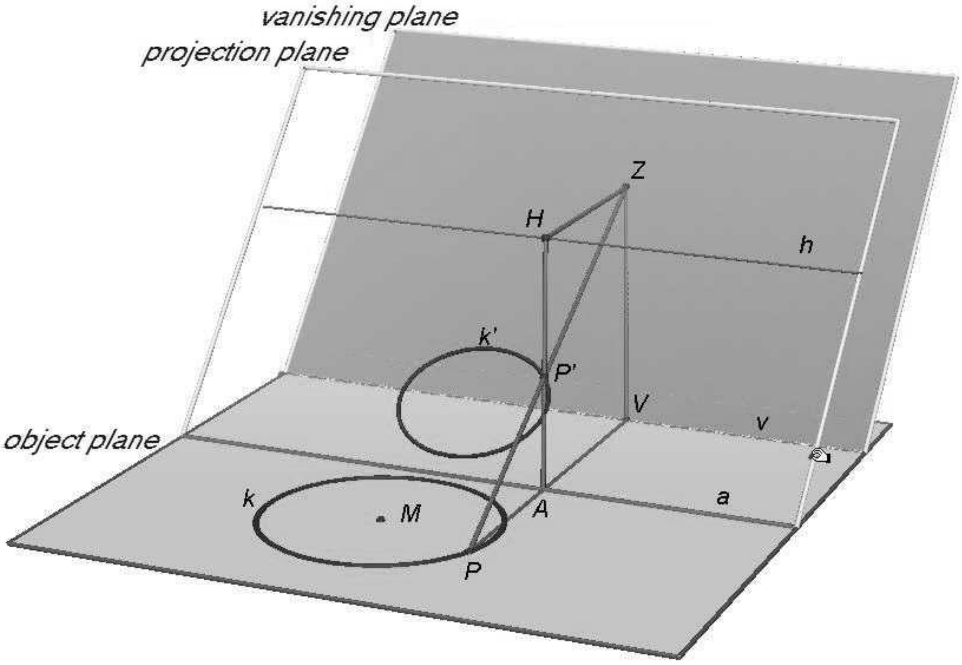

Figure 25: Figure 24 showing the tilting to the object plane

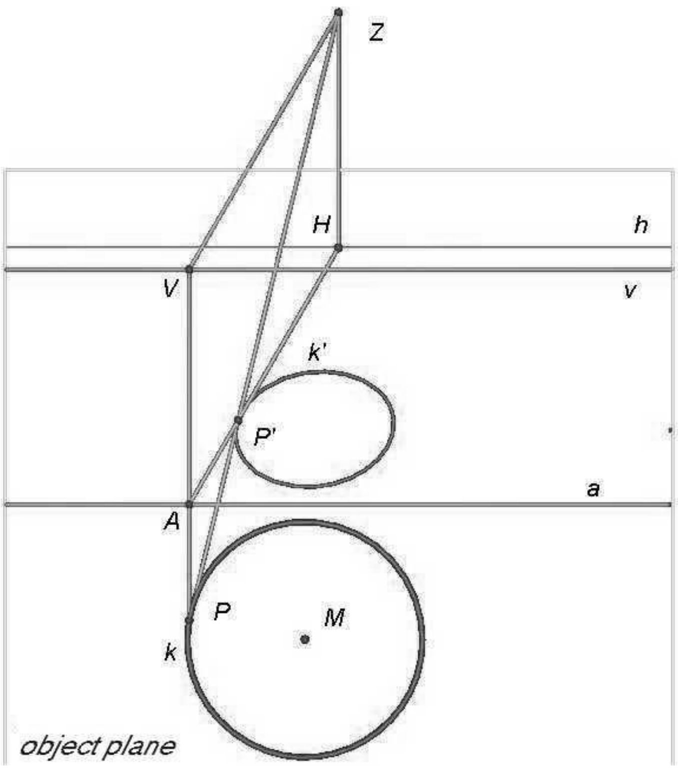

Figure 26: Final state of tilting: axial projective transformation in plane

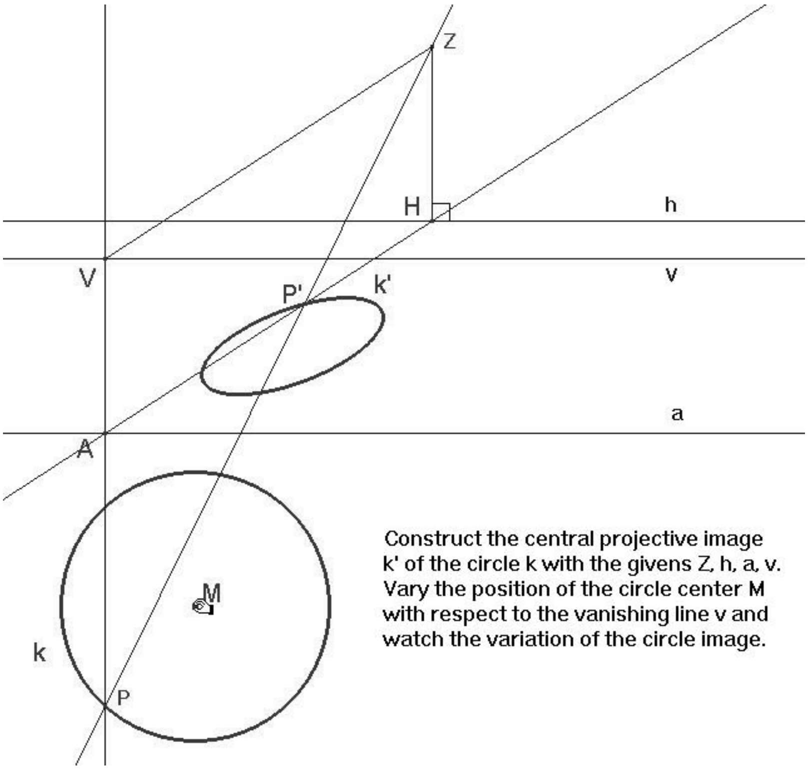

Figure 27: Construction of an ellipse as an axial projective image

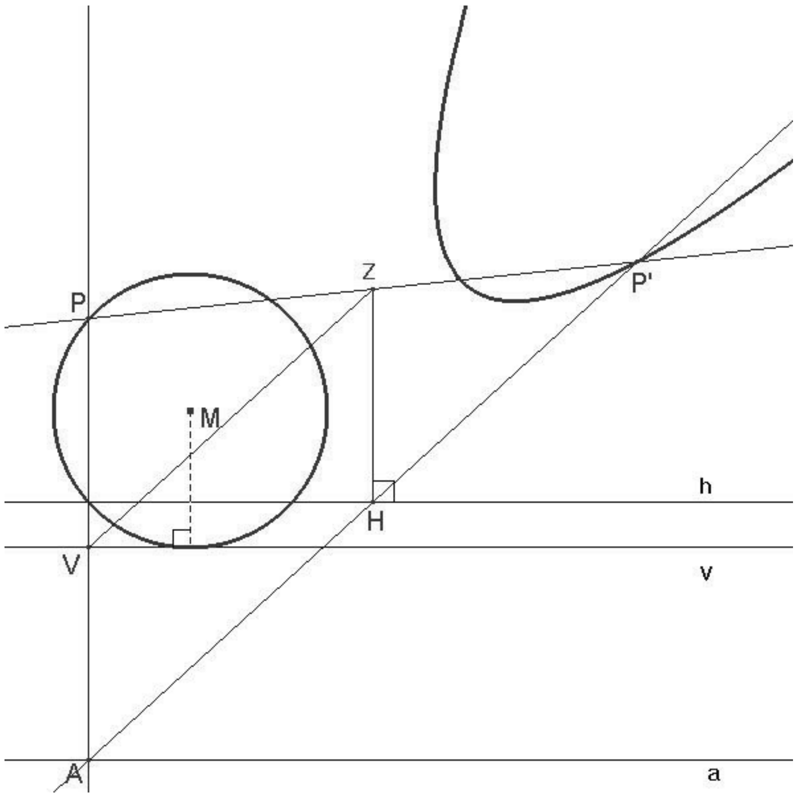

Figure 28: Construction of a parabola as an axial projective image

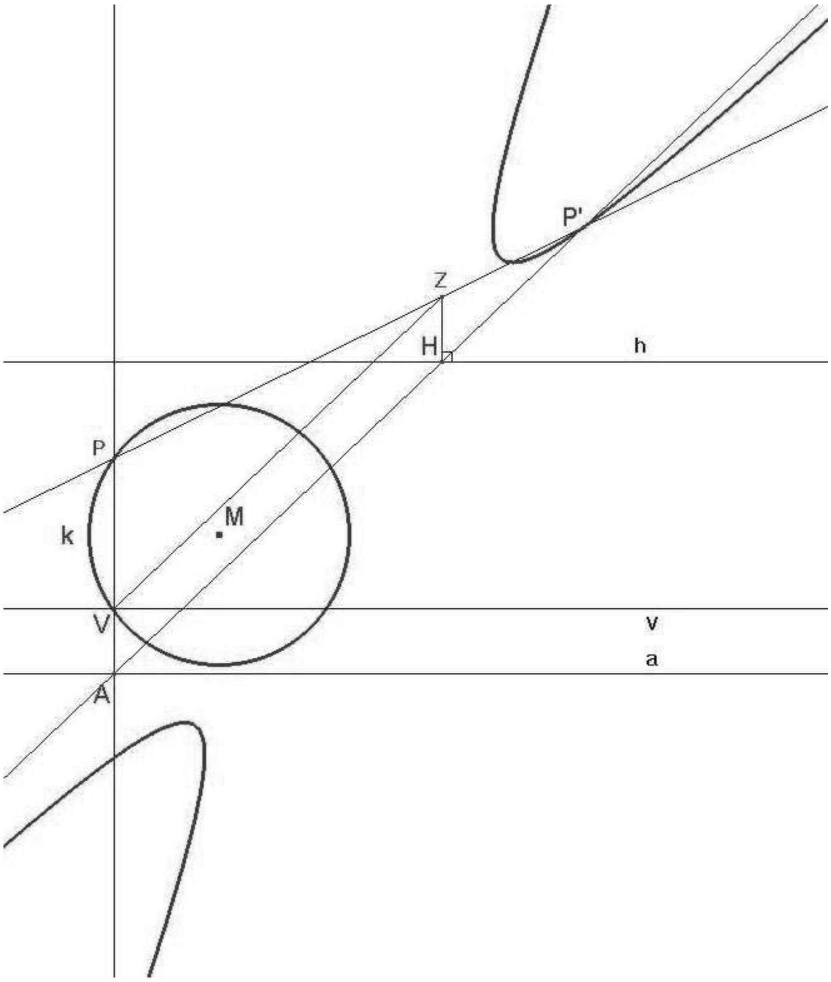

Figure 29: Construction of a hyperbola as an axial projective image

4. Conclusion

In similar way we are able to deal with all the phenomenal approaches to spatial conics in virtual space using Cabri 3D, which are a prerequisite knowledge to a further-reaching treatment.

For example, we can also think of conics as “intersections” of a cone of light with a plane. The curves formed with planes by the light cone going out from a point source Z (Figure 30) can be interpreted as images in central projection of a circle k. The circle plane is vertical to the cone axis on which the circle centre is located. With the aid of a rotating plane, Cabri 3D enables us to observe the change of shape of the curve from an ellipse into a hyperbola.

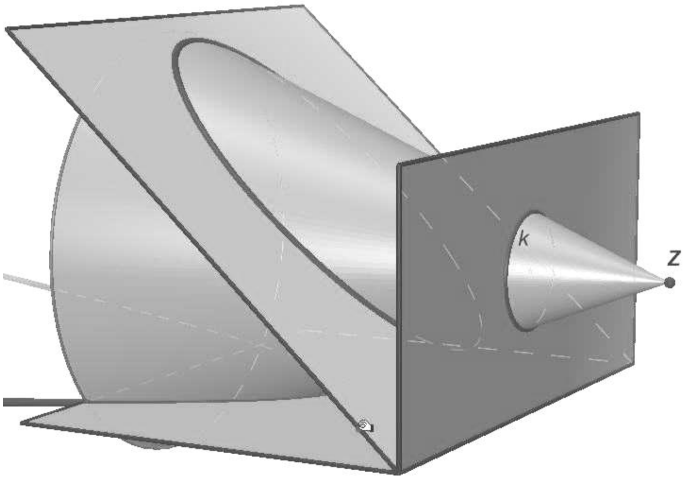

Figure 30: Cone of light intersected by planes

References

[1] Laborde, J. M. (2004). *20 Years of Cabri! Perspective of Geometry Based Computing Means of Tomorrow.* (presentation), Third Cabri Geometry International Conference, Rome 9th – 12th September 2004, Universitá "La Sapienza"

[2] Laborde, J. M.; Bainville, E. (2004). *Cabri Géomètre 3D* (Version 1.0). Grenoble: Cabrilog (www.cabri.com).

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