

Reconstructive Modelling inside Dynamic Geometry Systems

*In honour of Jean-Marie LABORDE,
the creator of Cabri Géomètre, on
the occasion of his 60th birthday.*

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1. Introduction

It is undisputed that non-mathematical applications, including mathematical modelling, in the mathematics classroom of general schools are an important argument in legitimating the teaching of mathematics.

For example, Corbitt and Edwards, in the NCTM Year-book 1979 “*Applications in school mathematics*”, describe the mathematical modelling process as follows: “The mathematical-modelling process can be summarized schematically, as shown. It involves (1) the formulation of a real-world problem in mathematical terms, that is, the construction of a mathematical model; (2) the analysis or solution of the resulting mathematical problem; and (3) the interpretation of the mathematical results in the context of the real-world problem.” (Diagram 1)

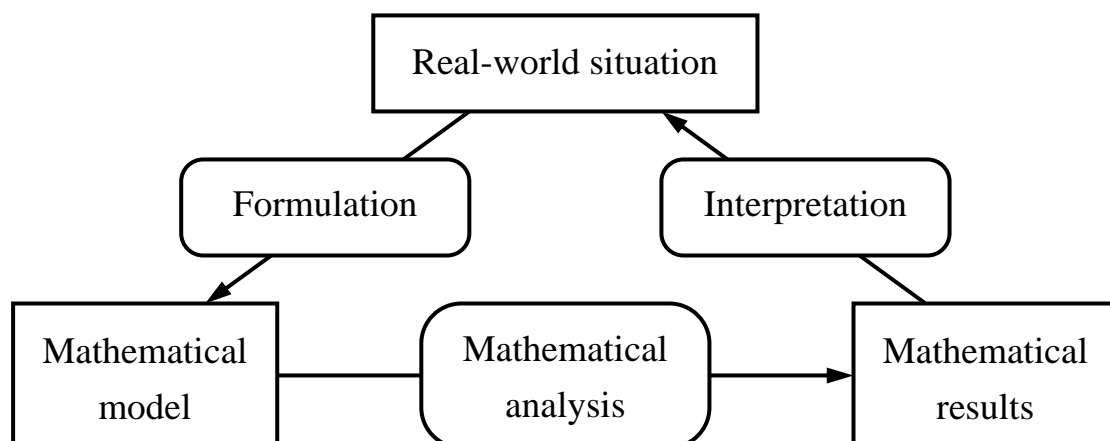


Diagram 1 (Mathematical modelling process)

The list of mathematical “*competencies*” (OECD 1999, p. 43) of the PISA

Framework includes modelling, which is defined in more detail in “*Modelling skill*”:

“This includes structuring the field or situation to be modelled; “mathematising” (translating “reality” into mathematical structures); “de-mathematising” (interpreting mathematical models in terms of “reality”); working with a mathematical model; validating the model; reflecting, analyzing and offering a critique of a model and its results; communicating about the model and its results (including the limitations of such results); and monitoring and controlling the modelling process.”

Geometrical modelling is a wide and interesting application of modelling, and elementary geometry is a powerful and even interdisciplinary modelling tool (see, e.g. Cummins, J. et al. “*Geometry: Concepts and Applications*”, 2001).

The traditional media used for geometrical modelling are supplemented today by computer graphics tools (Diagram 2). In our opinion, only directly manipulative computer graphics tools like Dynamic Geometry Systems (DGS) should be recommended for secondary grade pupils.

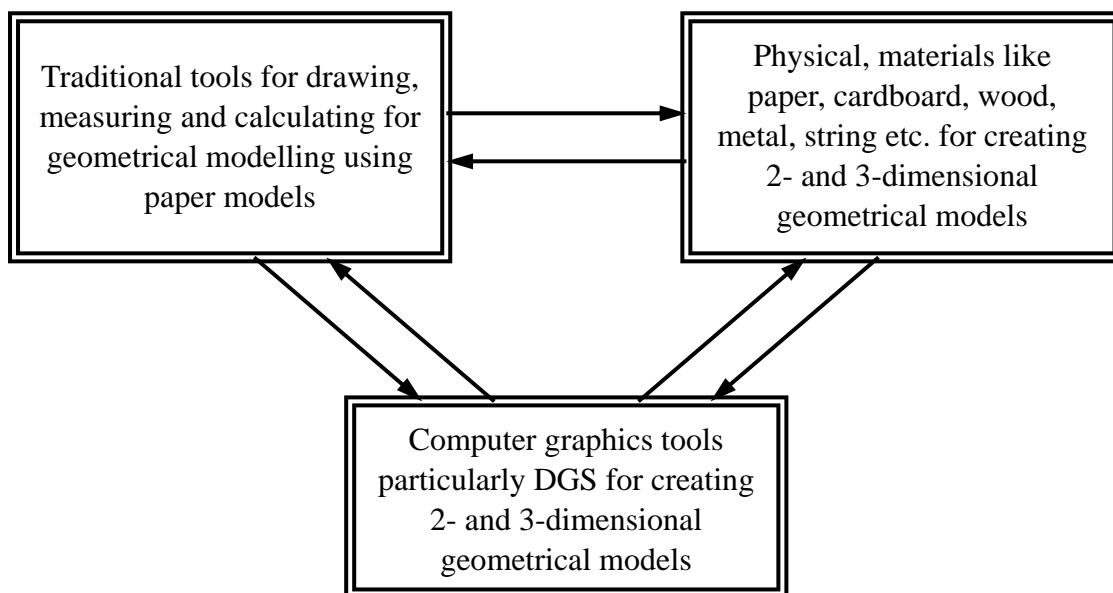


Diagram 2 (media-specific modelling)

Interface problems (see arrows in the diagram) between media-specific modelling options, in particular between computer-assisted and traditional

modelling in the geometry classroom, will not be considered here. (It is conceivable, as a contribution to media training in the geometry classroom, to introduce the currently available modelling media and to analyze their advantages and shortcomings.)

When using media-specific modelling aids and modelling tools, the following skill is required (OECD 1999, p. 49):

“Aids and tools skill. This includes knowing about, and being able to make use of, various aids and tools (including information technology tools) that may assist mathematical activity as well as knowing about the limitations of such aids and tools.”

Traditional media have the advantages of offering an integrative manner of perception and of training the coordination of visual and tactile activities, which is becoming increasingly important in view of the many hours children spend watching TV or playing with the computer mouse and keyboard. On the other hand, traditional modelling is limited in terms of the complexity, authenticity, up-dating, versatility, portability, reproduction, distribution, ease of communication and administration of the models created: the options for self-correcting; and the power of the modelling tools. Of course, sufficient skill in using the computer technology and computer tools is required if the advantages of the computer in the development and use of virtual geometrical models are to be fully utilized.

We are going to develop a method for geometrical modelling on the basis of DGS which intends to strengthen the part of geometrical construction in modelling (Schumann 2003). In our opinion, this application of the computer is an example of its effective integration in geometry teaching. DGS are powerful tools for modelling what can be interpreted as two-dimensional elements of our everyday environment. Technically, tools must be capable of importing image files of such elements into DGS. The imported images can then be reconstructed by modelling making use of the adequate characteristics of DGS, which offer far wider options than conventional modelling tools. Solutions to authentic modelling problems consist of the reconstruction of the geometry of elements found in nature or created by man, either intentionally or

unintentionally. As a result of the reconstruction process, we obtain static or dynamic virtual models.

Diagram 3 presents an outline of methods and ways of working with DGS in the context of geometry teaching in lower and middle secondary schools (Schumann 2001); modelling in DGS is supported by all other methods and options listed in the diagram.

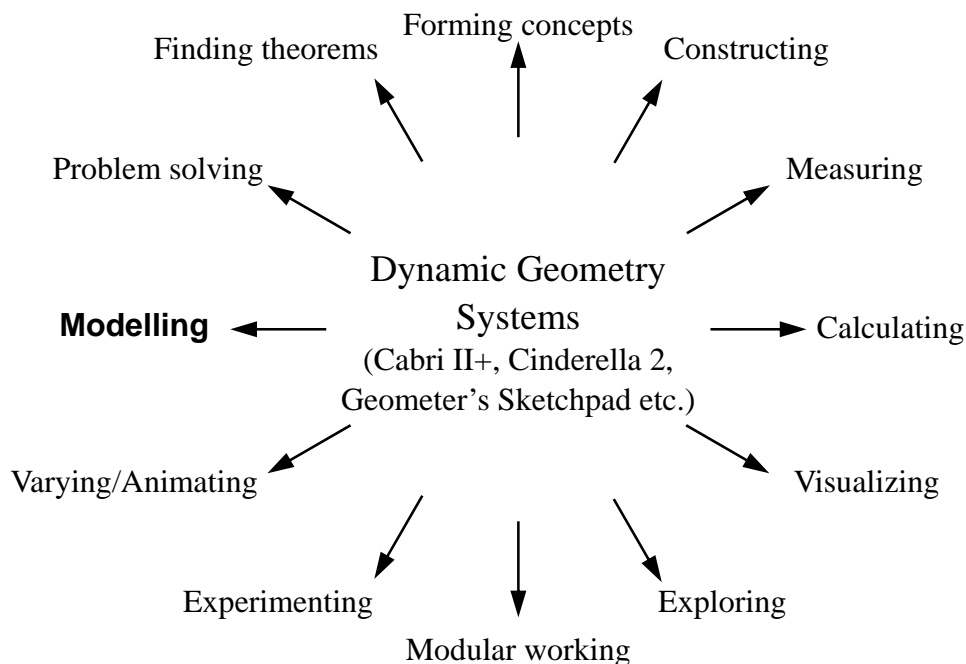


Diagram 3 (methods and ways of working with DGS)

When using DGS in geometrical modelling, we strive for the following general teaching goals in geometry:

- Training of geometrical perception, training the “geometrical eye” (teaching goal concerning the phenomenology of perception)
- Appreciating the usefulness of geometry (affective teaching goal)
- Applying and enhancing knowledge in geometry: concepts, statements and methods (Cognitive teaching goal)
- Experimental exploration and analysis of phenomena, which can be geometrically modelled (meta-cognitive teaching goal)
- Training the use of DGS (technical teaching goal)

The process of modelling is illustrated in Diagram 4.

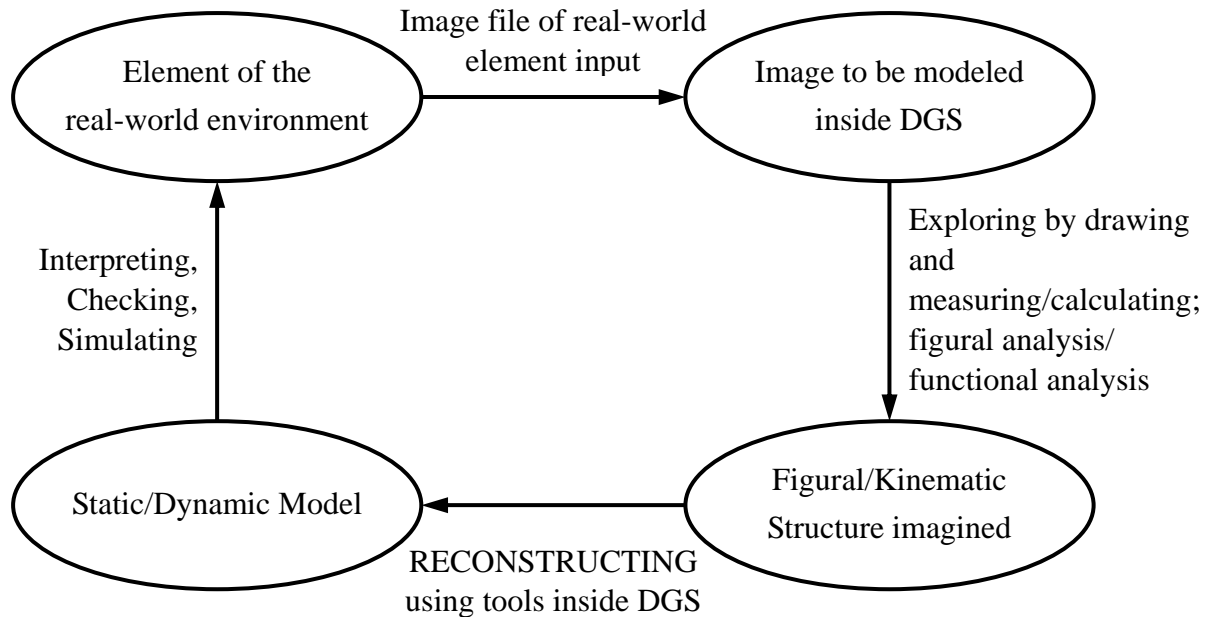


Diagram 4 (modelling process in DGS)

The modelling process, i.e. the creation of static and/or dynamic models, is described below in the form of a school project guideline.

Guideline for creating a static model:

1. Look for an object in your environment, in printed media or on the web which you think can be analyzed and reconstructed using the tools of 2-dimensional geometry.
2. Make a picture of that object with a digital camera, by scanning or by making a digital copy and load the image file into your DGS.
3. Analyze the picture of the object in DGS by drawing, measuring and calculating. Look for geometrical figures and rules. Attempt to find arguments for the rules.
4. Reconstruct the identified figure or configuration using the tools of the DGS.
5. Validate your reconstruction by comparing it with the original image.
6. Publish your verified model, together with a description and the picture of the original structure, e.g. on the web or in a poster.

Guideline for creating a dynamic model:

1. Look for a moving object in your environment, in printed media or on the

web which you think can be analyzed, reconstructed and simulated using the tools of geometry in the plane.

2. Make a picture of that object using a digital camera, a camcorder, by scanning or by making a digital copy, and load the image files or the digitized video into your DGS.
3. Analyze the picture/s of the moving object by drawing and measuring. Look for geometrical figures and rules. Observe the function of the moving object, or get information on its function. Attempt to find arguments for the rules and functions.
4. Reconstruct a functioning model using the tools of DGS.
5. Verify your reconstruction by means of simulations. Check whether the functionality of the reconstruction is sufficiently similar to the functionality of the original.
6. Publish your verified dynamic model, together with a description and the picture of the original object, e.g. on the web.

The description of mathematical modelling in the “PISA-Skills” applies, in principle, also to reconstructive modelling in DGS.

There is one major limitation to modelling in DGS: Only those objects can be modeled which can be described by the tools of 2-dimensional elementary geometry and which can be reconstructed using the tools and methods of DGS. (“Real” 3D modelling as a reconstruction method necessitates digital recording of a three-dimensional object or structure. This type of digitalization technology is not available for school geometry, and there are no 3D tools for school geometry to support the import of such 3D images for modelling.)

Another problem is encountered in the fact that knowledge from outside school mathematics about the object to be modeled must be taken into account. Otherwise the result might be amateurish.

2. Examples of Reconstructive Modelling

Preliminary Remarks

We are now going to present some examples of how static or dynamic models are created for natural or man-made objects of different levels of complexity. The structures presented here were selected with a view to

motivating pupils to find their own examples. In all cases, teachers must be competent in helping pupils to select suitable objects or structures. For example, care must be taken in central projective photography of objects that the image plane is in parallel position to the object plane and the camera is focused on the center of the object.

With the exception of reproducibility and/or correction options of the DGS constructions in the repeat or undo mode, technical details of DGS utilization will be omitted from the documentation of our examples. The DGS used in these examples was Cabri-Geometrie II Plus (cabri.com); this software is well suited for reconstructive modelling with regard to intuitive user guidance and with regard to the available options.

All examples have interdisciplinary aspects or aspects relating to general knowledge. Although these can be mentioned only briefly here, they may be used for extending the project beyond the limits of geometry. The objects selected for reconstructive modelling may come from different cultural contexts of local or universal importance.

Not all of the examples will conform to teaching schedules, but in our opinion the geometric description tools must be adapted to the real-world phenomena, not vice versa. Of course, this raises the problem of situative availability of descriptive tools and the limits of the computer software used.

2.1 Static Models

Example 1 (Flag)

Reconstruction of the design of a flag is an example of textile design reconstruction in general (e.g. also of quilts and patchwork). It is a motivating project subject for grades 7 and 8. Company logos, traffic signs etc. are suited as well.

The rectangular flag to be reconstructed in this example has a relatively simple design (Figure 1), with the length and width in a ratio of $2 : 1$, with an isosceles triangle whose height equals one third of the flag length, and with three horizontal stripes of equal width. The points of the pattern are reconstructed by constructing. It is not difficult to construct the isosceles triangle and the two congruent trapezoids as objects for coloring. An adequate

construction macro can be defined in order to enable reproduction independent of size.

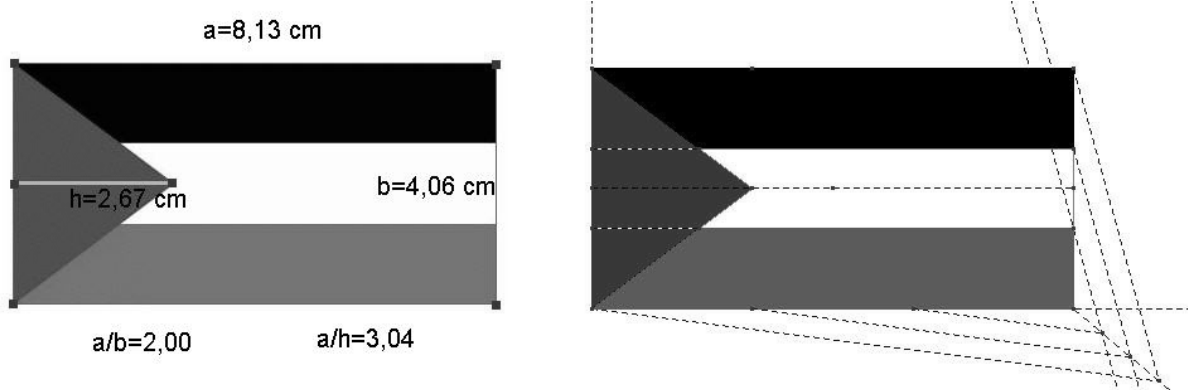


Figure 1

Example 2 (Painting)

As an example of the many works of art with a geometrical structure, we selected one of the “*Compositions with Red, Yellow and Blue*” (Figure 2, original size: 51cm x 51cm) by Piet Mondrian (1872 – 1944), all of which are constructed in accordance with geometrical proportions and according to the laws of perceptual aesthetics. For example, the picture contains a rectangle whose height is slightly less than its width, and which is viewed as a square by the human eye.

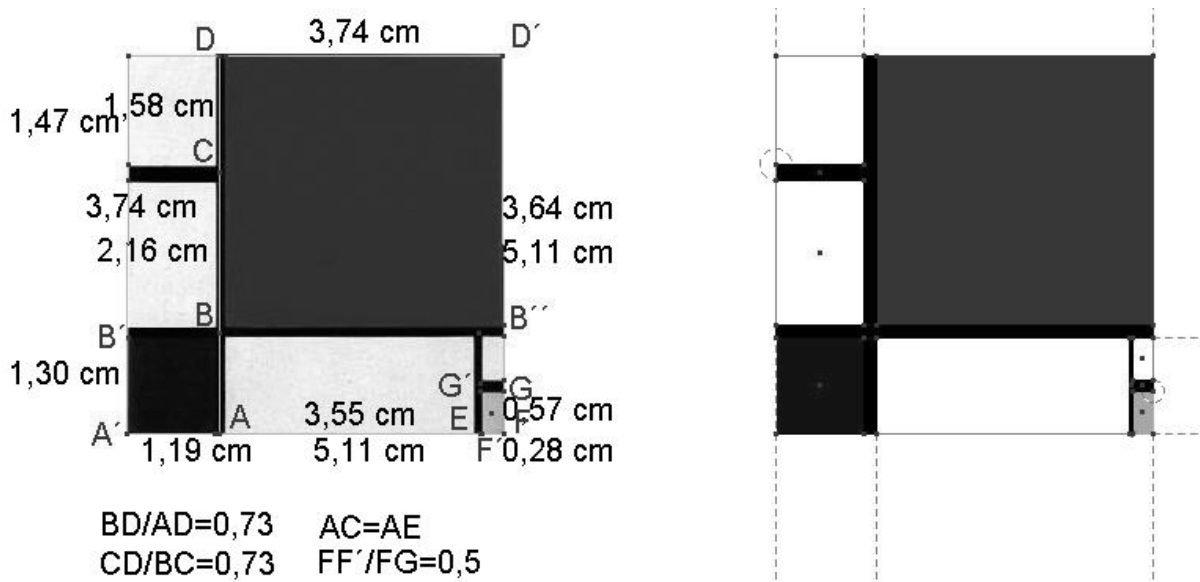


Figure 2

For a most economical reconstruction, one should start by drawing the red and blue near-squares, followed by the yellow half-square, etc.

Example 3 (Floor tile)

Concrete floor tiles are an interesting subject for reconstructive modelling and a realistic approach to the geometrical problems of tiling the plane.

In our example (Figure 3), a circumscriptive square can be constructed around the point-symmetrical “tile”. Using the reconstructed tile, the tiled floor can then be reconstructed by translation, line reflection and 45° rotations.

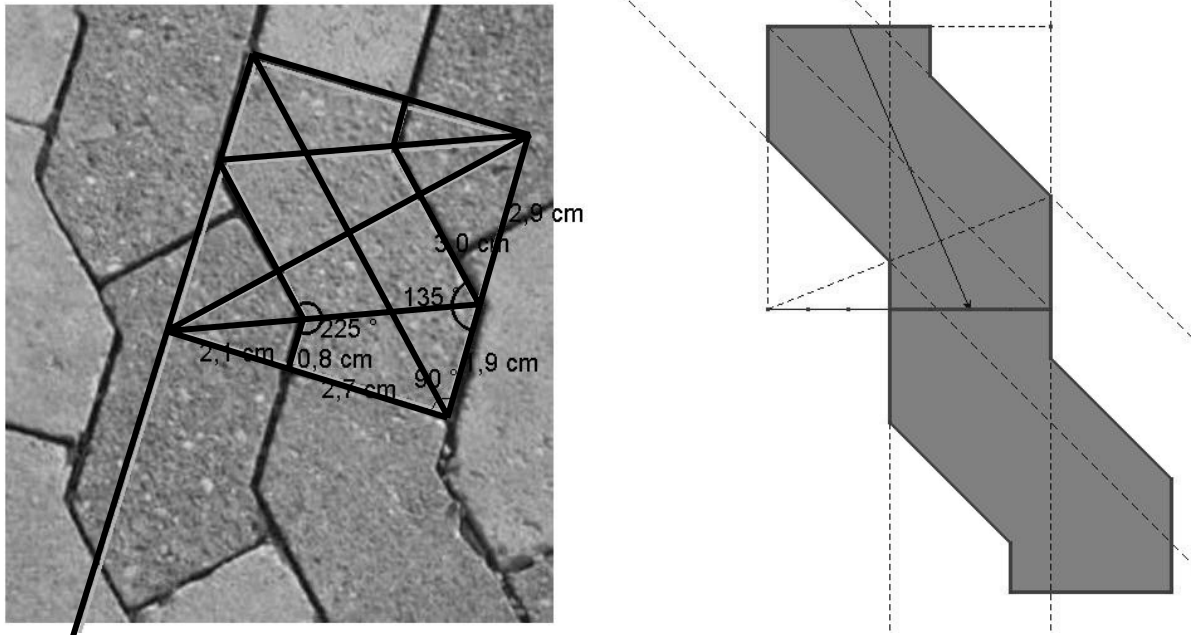


Figure 3

Example 4 (Building façade)

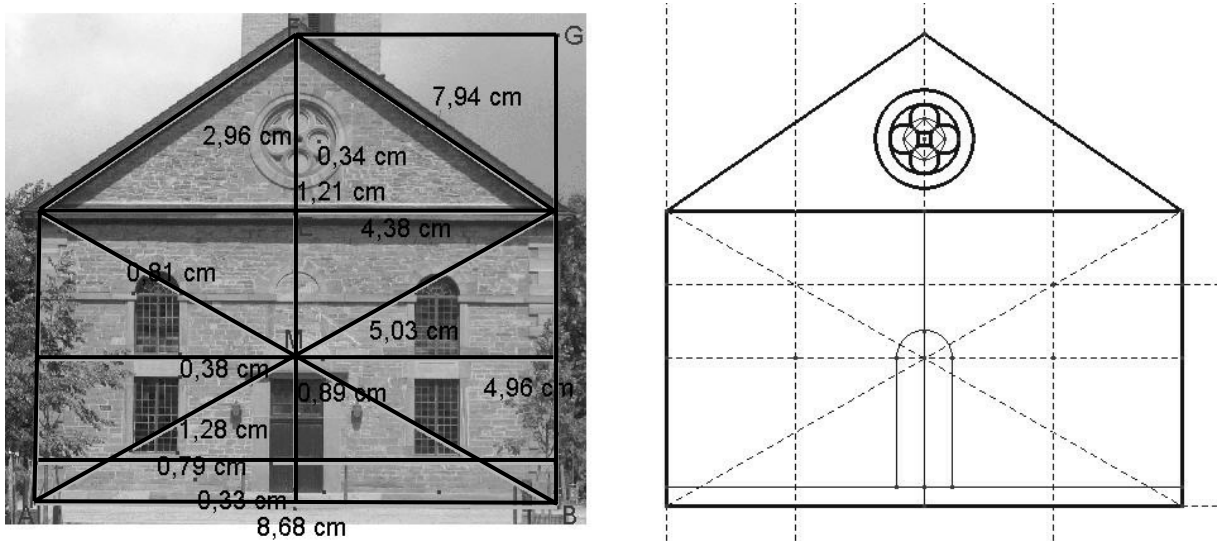


Figure 4

Building façades offer a wealth of reconstruction problems, and it is interesting to find out about the proportions used by the architect.

Our example (Figure 4) is a neoclassicist façade (church building at Birkenfeld near Pforzheim) with classic proportions. The rectangle ABCD is a so-called Sixton whose diagonals form two equilateral triangles (BCM, ADM). The frieze consists of two halves of a so-called Hemidiagon, i.e. rectangles with a ratio of $2 : 3$. Both types of rectangles are listed among the aesthetically pleasing rectangles (see, e.g. Wersin 1956). It was our intention here to find out about the geometrical proportions; it would take too long to reconstruct the entire façade.

Example 5 (Window)

Reconstruction of windows or doors can be a modelling challenge. In our example of a neo-gothic window (Fig. 5, ev. Stadtkirche, Weingarten/Württ.), the radius of the small lancet arch has a ratio of $3 : 1$ to the radius of the circle circumscribing the rosette. This is also the ratio of the circumscribing circle to the radius of the small concentric circle inside the rosette.

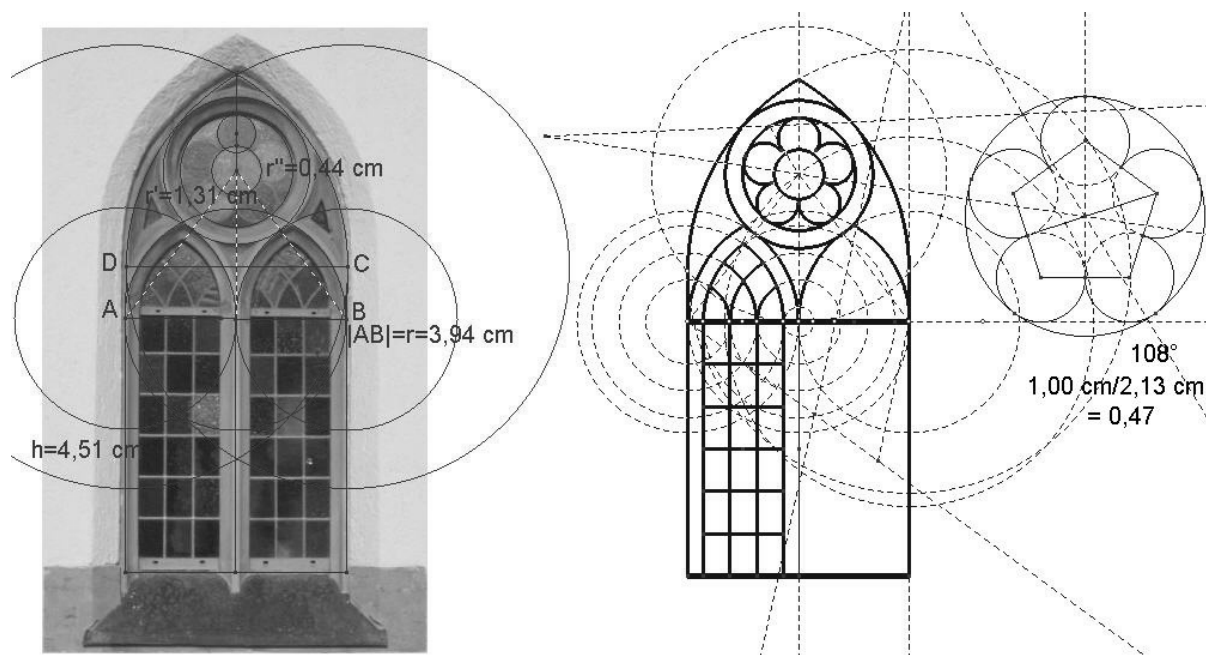


Figure 5

To reconstruct the big lancet arch comprising the upper part of the window, one must proceed as follows: There is a circuit with a center point and the radius of a second circle with a straight line as the geometric locus of its center point. Construct the latter so that the two circles touch. – The rosette window is constructed by means of a suitable dilation.

This type of reconstruction problem is suited for a teaching project in grade 9.

So far, all examples had polygonal and circular shapes only. We are now going to proceed to more difficult shapes, e.g. algebraic and transcendental curves. This necessitates greater mathematical skills and more powerful tools for modelling in secondary education. The dynamic adaptation of the geometric object to the image of the real object relies more strongly on direct manipulation and on the drag mode. Further, plotting of curves, e.g. in the form of polar coordinates, will support the modelling process.

Example 6 (Ground plan)

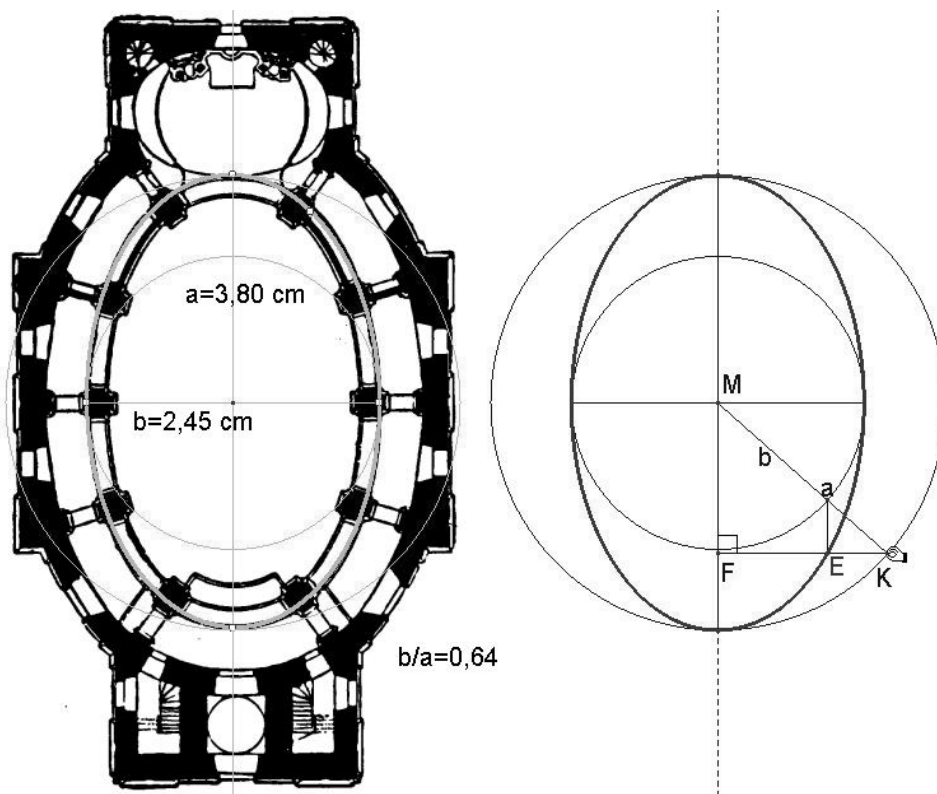


Figure 6

Elliptic shapes are found in many objects of our everyday environment, e.g. in baroque buildings. Our example is a ground plan of the baroque church at Steinhausen (Figure 6). The modelling process starts by drawing the two main axes of the picture, whose ratio approximately reflects the golden section. Using a fifth point on the circumference of the ground plan, we construct an ellipse through the five points which reflects the shape of the ground plan. We reconstruct the ellipse as an image of a circle by means of orthogonal affinity as usual. To verify our result, we draw it as an overlay over the original image

(no picture given). – This can be followed by an analytical geometrical description of the ellipse.

Example 7 (Vault of a bridge)

For static reasons, bridge vaults are parabolas which can be described, e.g., by an equation of the type $y = ax^2$. To model the Millennium Bridge in Newcastle (Figure 7), we drag the origin of a Cartesian coordinate system into the apex of the bridge vault together with a parabola and its equation. After this, the value of a is varied per slider until the parabola is congruent with the bridge vault. The resulting equation is $y = -0.17x^2$; and $y = -0.02x^2$ for the lower pedestrian bridge (Schumann 2001). – It is now possible to reconstruct the vaults within a coordinate system.

Alternatively, we could construct a parabola as a local line using straightedge and compass, followed by dynamic adaptation of that parabola to the real bridge vault by changing the position of the focus.

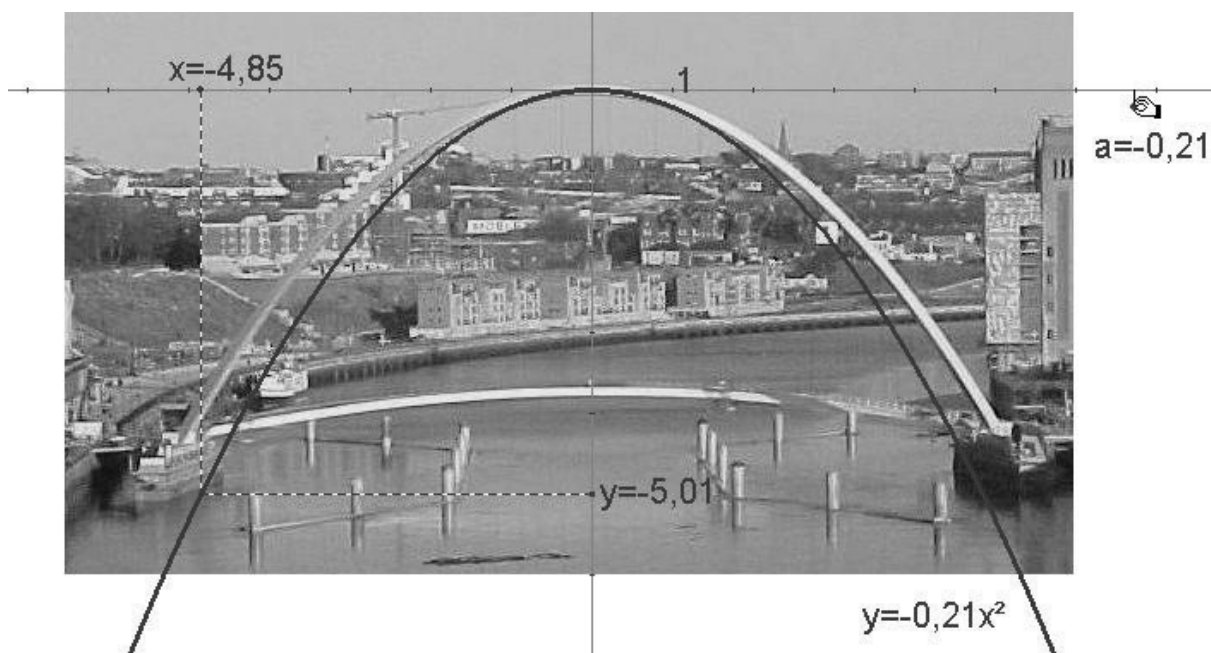


Figure 7

Example 8 (Rosette)

This symmetrical twelve-petal rosette adorns a designer-made bathroom tile (Figure 8). We are going to reconstruct it on the basis of the polar coordinate representation of the rosette curve $r = a \cos(b\alpha)$, in which a is the rosette size and b (b is even in our example) is half the number of petals. A separate drawing is used here to get a clearer picture. The values of a and b are varied using sliders until the curve has the correct shape. To

verify the result, the origin of the coordinate system is shifted into the center point of the rosette. (We assumed a slightly higher value of b to get wider contours; as a result, the petals in the resulting image are closer together.)

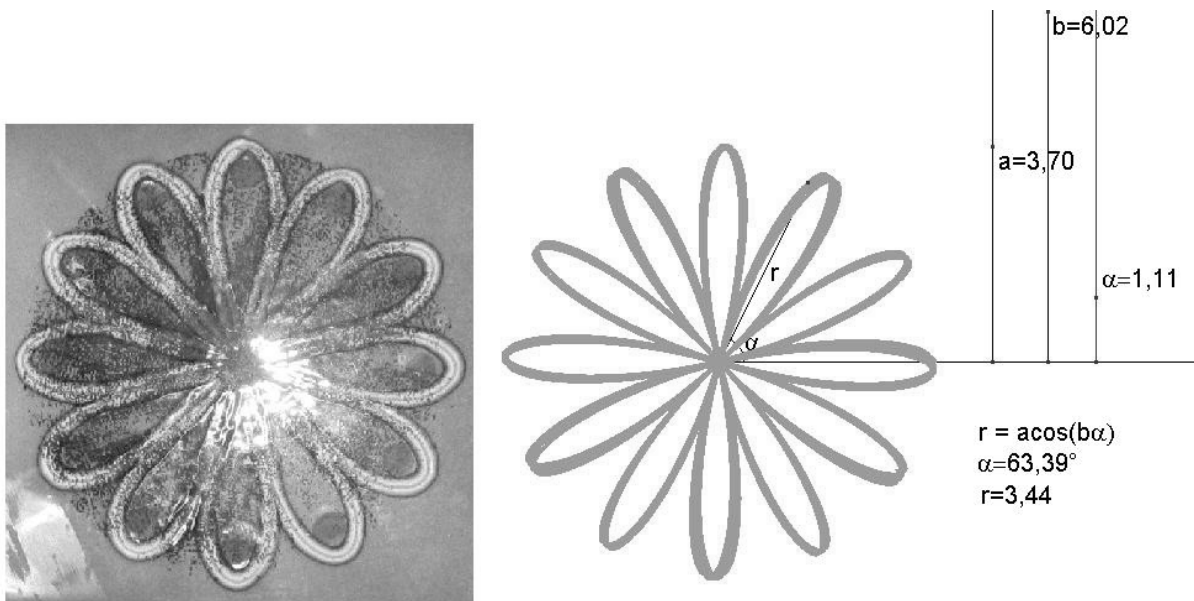


Figure 8

Flowers with radically symmetrical petals can be modeled by equally sectioning the circumference a circle, drawing circles around these equidistant points and modelling the leaves by circular arcs.

Example 9 (Snail)

The snail of Nautilus (Figure 9) has a spiral contour. By experimentation on the model image, we find that all rays which originate in the center point intersect the contour of the snail in the same angle (about 79°). This type of spiral is known as a logarithmic spiral with the following polar coordinate equation: $\rho = a e^{\frac{\psi}{\tan \alpha}}$, in which the parameter a is the size of the spiral and α is the angle of intersection as described above. The logarithmic spiral is centred and adapted to the contour using corresponding sliders for varying a and α .

This shape, i.e. the arc of a logarithmic spiral, is also found in paper-cutting knives; this shape keeps the angle between the knife and the paper constant, thus ensuring optimum pressure and cutting performance.

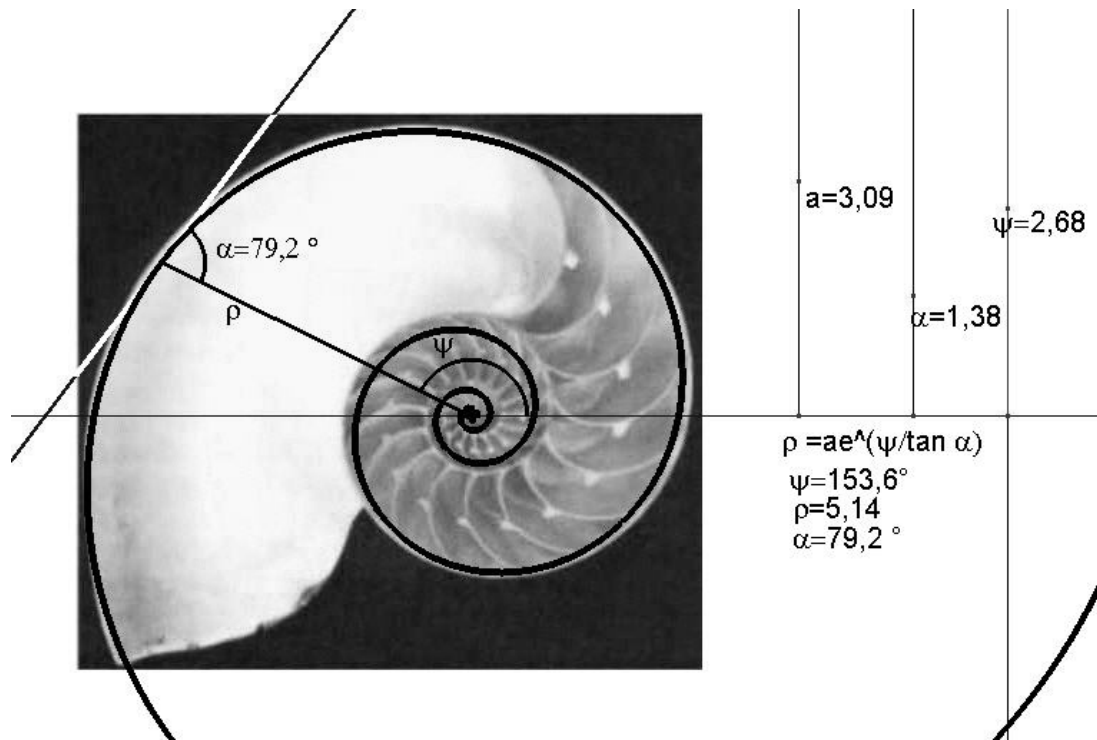


Figure 9

Example 10 (Butterfly)

Geometrical modelling of the axially symmetrical contour of a butterfly (Figure 10, *Argynnis paphia*) ideally requires option for drawing a “well-interpolating” curve for selected key points. Our DGS does not provide this option. We therefore used a polygon with many corner points that describe one half of the butterfly contour. We then created a flip-image of this polygon, and the resulting axially symmetrical model is shown on the right after canceling the picture.

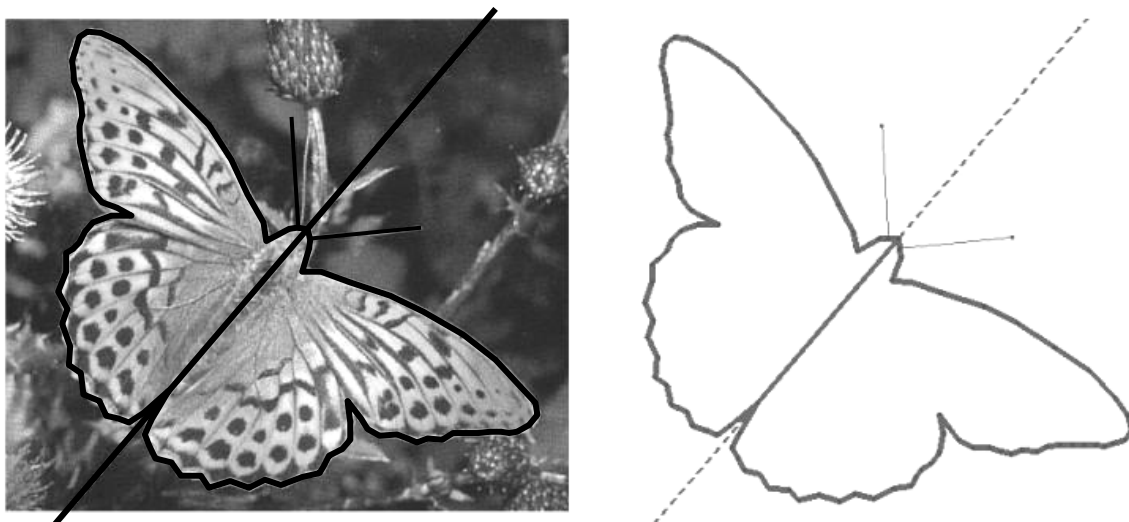


Figure 10

2.2 Dynamic Models

Models constructed with DGS are simplified models simulating the main function of a movable object. Functional testing and simulation of a dynamic model is carried out either manually in drag mode or by automatic animation (which has the advantage of keeping the observer undistracted). The track mode is used for (static) documentation of the different phases of the movement.

Dynamic DGS models were first presented in “Discovering Geometry with a Computer” (Schumann & Green 1994), although without authentic model images.

A wealth of reconstructions of moving elements and machines in simple dynamic models is found in Brian Bolt’s book “Mathematics meets Technology” (1991). Clearly, there is an interdependence between geometry, engineering and physics (see also Romanovskis/Schumann 2002).

Example 11 (Windmill)

A very simple dynamic model can be constructed for a windmill (a windmill in Andalusia, figure 11). The wheel is of decagon (10-gon) structure. Pulling it to the picture, it becomes alive by animated rotation.

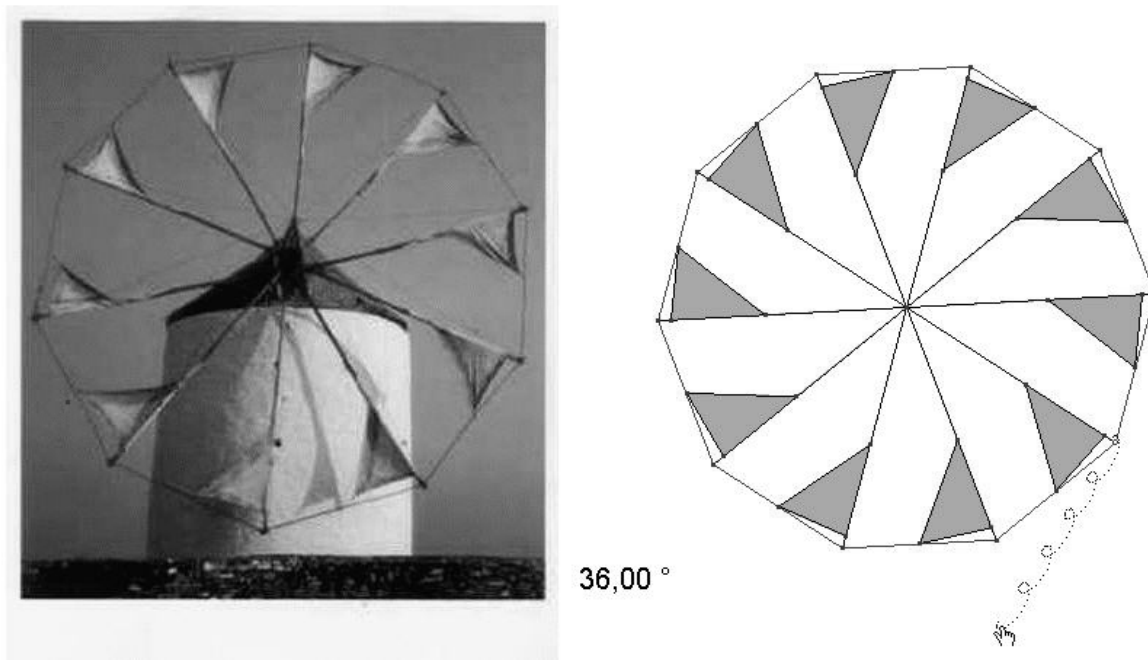


Figure 11

Example 12 (Garage door)

Up-and-over garage doors can be reconstructed by dynamic models. (Garage doors are designed with a view to minimum space requirements during opening and closing.) A simplified reconstruction from side view is not difficult. Figure 12 shows a phase image of the movement of an up-and-over garage door. The side view gives us an envelope – can you identify it?

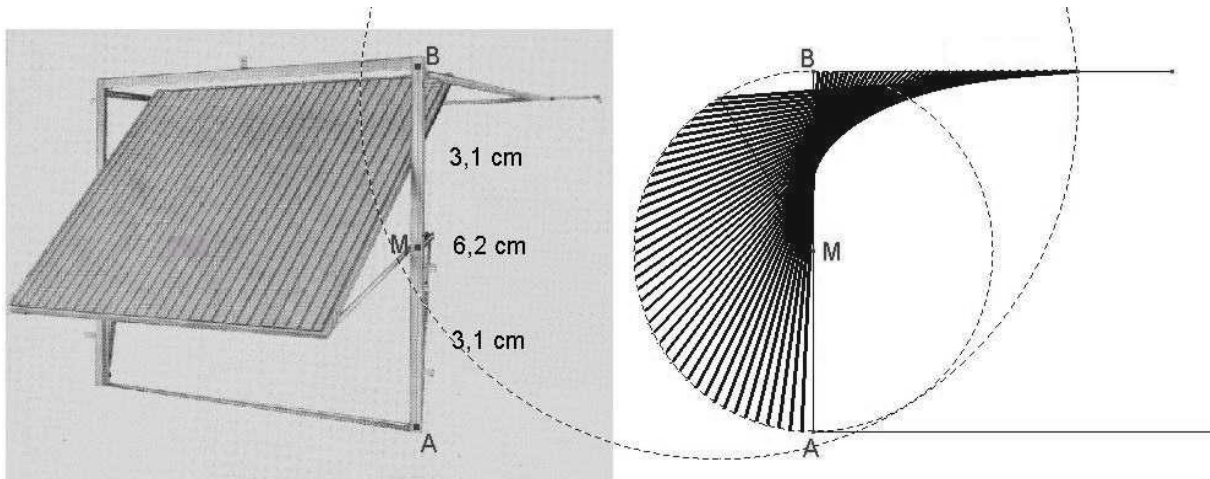


Figure 12

Example 13 (Platform lift)

This four-bar mechanism is found in many kinematic applications: Portable fence, pot stands, extendable lamp etc. Our example is a platform lift (Figure 13), which is easy to reconstruct using images of point and line reflections. – Can you identify the loci along which the outer joints of the platform lift move?

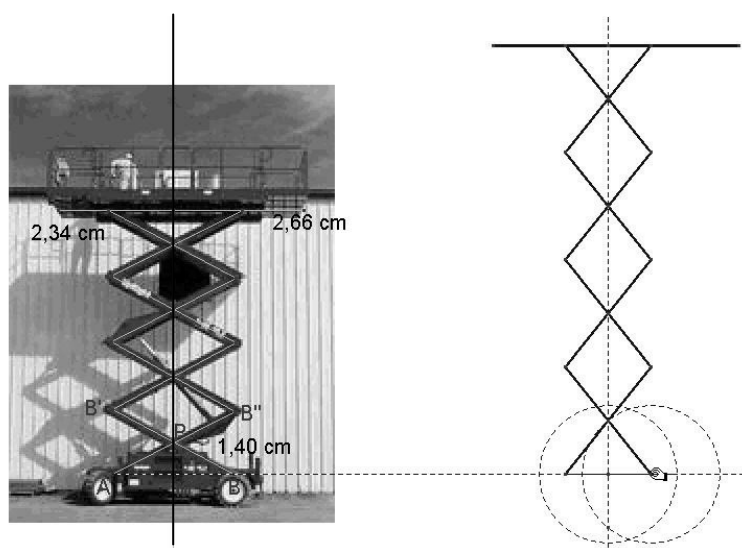


Figure 13

Example 14 (Engraving machine)

Mechanical engraving machines (see the example in figure 14) are so-called pantographs, i.e. instruments for drawing reduced or enlarged copies of pictures.

As P follows the contours of the model image (e.g. a template) P' generates a reduced similar image. Our analysis shows a dilation with the center point E . The scale of reduction is defined by the position of P' on AB .

The four-bar mechanism $ABCD$ is a lozenge with a side length of 3 units; side CP respectively AE are assumed to have a length of 2 units respectively one unit.

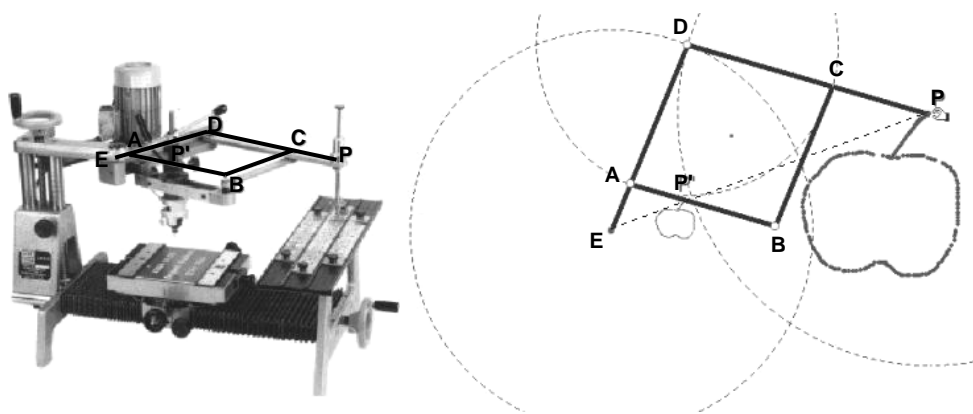


Figure 14

Example 15 (Slewing crane)

The crane in figure 15 consists, in principle, of the four-bar mechanism $ABCD$ with a fixed side AB (gantry) and the jib-head member DE . The modelling process leads to a simplified reconstruction with C as the moving point of the jib-head member. – The load to be transported by the crane must remain at a given height after lifting in order to prevent dangerous vibrations, so the jib head can move only horizontally. For this reason, the four-bar mechanism and the boom are dimensioned so that the straight line through point E and the point of intersection F of the extended AD and BC is vertical in the operating range (Stührmann; Wessels 1981). The nearly horizontal movement of E is indicated by the appropriate section of

the local line, an algebraic curve of the third degree which is generated by E as the boom moves.

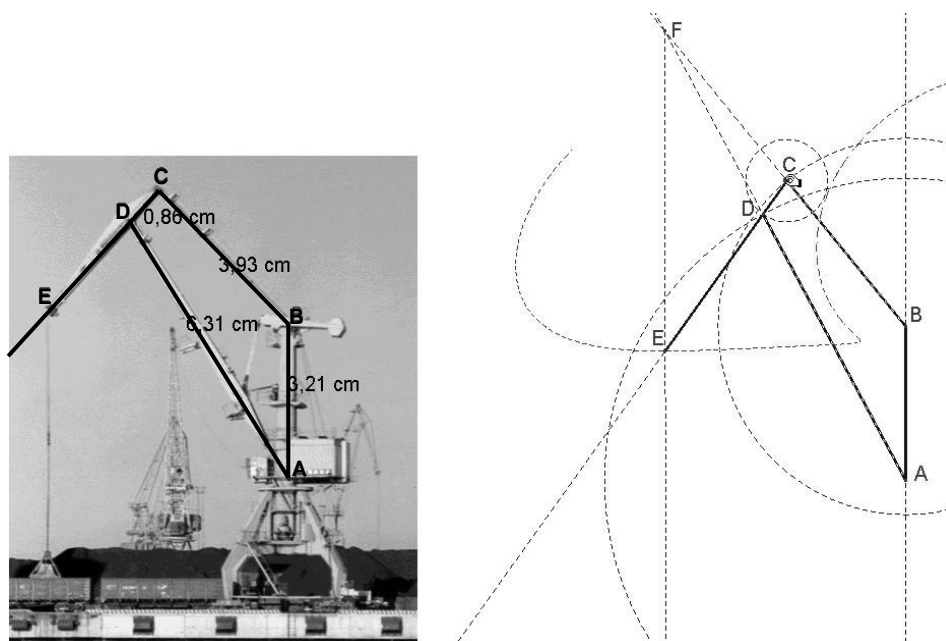


Figure 15

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