# On the sailors and the coconuts problem

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## 1. Introduction

On 9 October 1926, Ben Ames Williams posed a problem called "The sailors and the coconuts problem" on the Saturday Evening Post and aroused a lot of interest. It is a recreational problem and its origin can be traced back to the ancient India (Rothery, 1995). Its modified versions can be found in mathematics textbooks and literature nowadays, such as "The women and the diamonds problem" (Krantz, 1997, p.167) and the "The monkeys and the bananas problem" (Gannon et al, 2001). There are various ways for solving this problem, namely the Euclidean algorithm, the technique of generating function and the dynamical systems approach (Gannon et al, 2001), etc. In this article, we present a simple approach to tackle this problem, as well as its generalized version, using elementary mathematics only.

## 2. The problem and its solution

The sailors and the coconuts problem can be described as follows:

"Three sailors spent one day to gather coconuts and then decided to share them in the next day. In the mid-night, one sailor awoke quietly and tried to divide the coconuts into three equal shares. When he did so, there was one coconut left and he threw it away. He hid one share and then returned to sleep. After a while, another sailor awoke and he also tried to divide the coconuts into three equal shares. Again, there was one coconut left and he threw it away. He hid one share and then returned to sleep. Later, the third sailor also awoke and tried to divide the coconuts into three equal shares. Once again, there was one coconut left and he threw it away. He hid his share and then returned to sleep. When the three sailors got up in the next day, they started to divide the remaining coconuts into three equal shares and threw away one coconut that was left over. What is the least possible number of coconuts gathered by the sailors at the beginning?"

This problem can be formulated by a system of equations as follows:

$$\begin{cases} n = 3a + 1 \\ 2a = 3b + 1 \\ 2b = 3c + 1 \\ 2c = 3d + 1 \end{cases},$$

where $n$ denotes the original number of coconuts, and $a + d$ , $b + d$ and $c + d$ denote the shares of the three sailors respectively, starting with the one who divided the coconuts first and so on. By eliminating $a$ , $b$ and $c$ , we can obtain $8n - 81d = 65$ . Using the Euclidean algorithm or otherwise, we can solve for the least positive integral value of $n$ , namely $n = 79$. Although this approach can be applied to similar problems with more than three sailors, the calculations involved will be quite tedious.

Now, we present another approach to tackle this problem. The crucial step is to rewrite the above equations as follows:

$$\begin{cases} n = 3a + 1 \\ 2(a+1) = 3(b+1) \\ 2(b+1) = 3(c+1) \\ 2(c+1) = 3(d+1) \end{cases}.$$

By using multiplications and a bit cancellation, we get $a + 1 = (\frac{3}{2})^3(d + 1)$ and $d + 1 = 8k$ , where $k$ is a positive integer. Hence, $a = 27k - 1$ and $n = 81k - 2$ . Therefore, the least possible value of $n$ is $79$ .

## 3. A generalized version of the problem and its solution

The approach discussed in the last section works equally well with a more generalized version of the sailors and the coconuts problem. For instance, let us assume that the number of sailors is $k\ (\geq 3)$ . Using similar notations as above, the problem can be formulated as follows:

$$\begin{cases} n = ka_1 + 1 \\ (k-1)a_1 = ka_2 + 1 \\ \vdots \\ (k-1)a_k = ka_{k+1} + 1 \end{cases},$$

where $n$ denotes the original number of coconuts and $a_1 + a_{k+1}$ , $a_2 + a_{k+1}$ , …… , $a_k + a_{k+1}$ denote the shares of the sailors, starting with the one who divided the coconuts first and so on. Now, let us rewrite the system of equations as:

$$\begin{cases} n = ka_1 + 1 \\ (k-1)(a_1 + 1) = k(a_2 + 1) \\ \vdots \\ (k-1)(a_k + 1) = k(a_{k+1} + 1) \end{cases}.$$

By using multiplications and a bit cancellation, we get $a_1 + 1 = (\frac{k}{k-1})^k(a_{k+1} + 1)$ and $a_{k+1} + 1 = (k-1)^k w$ , where $w$ is a positive integer. Hence, $a_1 = k^k w - 1$ and $n = k^{k+1} w - (k-1)$ . Therefore, the least possible value of $n$ is $k^{k+1} - (k-1)$ .

## 4. Conclusion

While Gannon and Martelli (2001) suggested an interesting approach for solving the sailors and the coconuts problem by using the dynamical system approach, the method presented in this paper is simpler and more accessible to students with the knowledge of elementary mathematics only. In addition, this approach can be adopted to handle a more generalized version of the problem with more than three sailors with surprising ease.

## References

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2. Krantz and Steven, (1997). *Techniques of problem solving*. American Mathematical Society.
3. Rothery, (1995). *Hindu mathematics to computer algebra*. Micromath, 11(1), 5 – 7.