Addition, Subtraction, Multiplication and Division of Fractions: Combining intuitive understanding with efficient computation by cutting food*

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Introduction

Should we teach for student understanding or for mastery of procedures? Advocates of intuitive understanding claim that focusing on mathematical manipulation encourages students to compute blindly. On the other hand, detractors charge that class activities promoting understanding are divorced from the computations, thereby hindering students' algorithmic mastery. In this article, I argue that the "understanding vs. procedures debate" presents a false dichotomy. Using food metaphors that are sufficiently isomorphic to mathematical relationships, I present activities for teaching fractions that combine intuitive understanding with efficient computation, thereby creating a system of mathematical relationships that encompass both.

Through these activities, students can a) connect understanding with problem solving and b) create general, applicable procedures. Building on students' understanding of cutting food, discuss fractions as parts of square units of food (waffles, brownies, etc.). Then, introduce analogous and general cutting procedures to perform fraction computations. This article covers multiplication, addition, subtraction, and division of fractions.

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Multiplication

Introduce multiplication of fractions through cutting and re-cutting pieces of food.

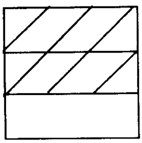
T: Let's say this is a waffle. [Draws a square on the blackboard.] If Ana wants two-thirds of it [writes "$\frac{2}{3}$"]. What would you do?

S: Cut two-thirds of it and give it to her.

T: Show me. [hands the student a ruler and a piece of chalk]

[S draws a horizontal line two-thirds from the top, and points to the upper part.]

T: Draw in a dotted line to show the three equal pieces [S draws the line] OK, so she cuts the waffle into three total pieces, [circles the "3" in "$\frac{2}{3}$"] and wants two of them, [circles the "2" in "$\frac{2}{3}$" and marks the two desired parts with diagonal lines. See figure 1.] We can also write this as two-thirds times one equals two-thirds. [Writes "$\frac{2}{3} \times 1 = \frac{2}{3}$".] How much Ana wants, two-thirds, [points to $\frac{2}{3}$], of [points to x], the original waffle [points to 1].



Divide the waffle into three equal sections -> $\frac{2}{3}$ <- Then take two of them

$\frac{2}{3} \times 1 = \frac{2}{3}$

Figure 1. Introduce the fraction square by taking $\frac{2}{3}$ of a waffle. Divide the square into three equal horizontal sections and cover the upper two with diagonals.

After familiarizing the class with the *fraction square* diagram, continue by cutting a previously-cut sandwich.

T: Let's say Ben only wants three-fifths of a sandwich for lunch. [Draws a square] How would you draw that?

[S divides the sandwich into five rows and covers the three upper rows with diagonals. T writes "$\frac{3}{5}$" under S's diagram.]

T: OK. Now I'm going to rotate our two-thirds picture to help us solve the next problem. [Draws a square cut into three columns with two columns covered with opposing diagonals and writes "$\frac{2}{3}$"]. Ben brought his three-fifths of a sandwich to school but only ate two-thirds of it. How should we cut two-thirds of Ben's three-fifths sandwich to show the part of his original sandwich that he ate? (see figure 2).

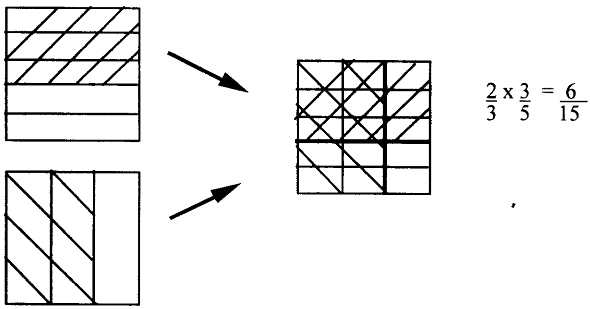


Figure 2. Calculating $\frac{2}{3} \times \frac{3}{5}$. After drawing fraction squares for $\frac{2}{3}$ and $\frac{3}{5}$, rotate the $\frac{2}{3}$ fraction square and place it on top of the $\frac{3}{5}$ fraction square. The number of cross-hatched sections (6) divided by the total number of sections in the solution fraction square (15) is the answer, $\frac{6}{15}$.

S: Cut the three-fifths sandwich into thirds and take the first two. [Draws three columns on to the three-fifths fraction square and covers two of them with opposing diagonals].

T: Now, we've drawn the two-thirds square on top of the three-fifths square. [Darkens the third vertical line and the fourth horizontal line] Because Ben only eats two-thirds [sweeps hand along left two-thirds] of his three-fifths sandwich [sweeps hand along upper three-fifths], he only eats this upper left corner [traces finger along upper left corner]. So how many pieces did he eat?

S: Six.

T: And how many total pieces of the final cut-up sandwich can you count?

S: Fifteen, so he ate six-fifteenths of the original sandwich.

T: Yes. We can also write two-thirds times three-fifths equals six-fifteenths. [Writes "$\frac{2}{3} \times \frac{3}{5} = \frac{6}{15}$"]. If we didn't have pictures, how could you compute two-thirds times three-fifths?

S: Multiply across.

T: Yes. Multiplication creates more pieces like cutting. The original sandwich was cut into five total pieces and Ben wanted three of them, three-fifths. When we cut it again, we cut the five total pieces [points to "5"] by three [points to "3" in "$\frac{2}{3}$"], we create three times five or fifteen total pieces [points to "15"]. Although the sandwich pieces that Ben wanted were also cut by three columns, Ben only ate two of them and left one behind. In effect, the pieces that Ben wanted originally were multiplied by two [points to "2"], so he ate two times three or six pieces.

After a few more square food problems, do problems that do not involve square objects and eventually move on to direct computations without objects. (Note: to multiply mixed numbers (for example, $1\frac{1}{3} \times 2\frac{1}{2}$), put one set of fraction squares on top of the other set (see figure 3).

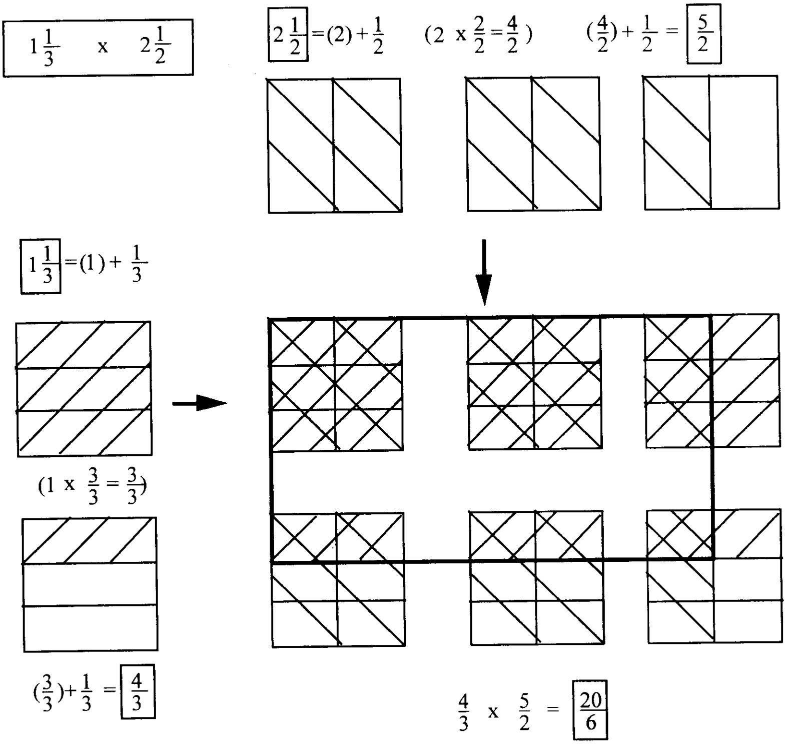


Figure 3. Multiplying mixed numbers, $1\frac{1}{3} \times 2\frac{1}{2}$. Draw the fraction component as before and draw the whole number component as whole squares cut into the number of sections specified by the fraction denominator. Draw the fraction squares of the first mixed number along a column and the second along a row. Then, place each column square on top of each row square. The number of cross-hatched sections (20) over the number of sections in one square (6) is the answer, $\frac{20}{6}$.

To make a fraction square for $\frac{A}{B}$

- o Cut the square into B equal rows
- o Mark A of the rows with diagonals

To multiply two fractions,

- o Create a fraction square for each fraction
- o Rotate one fraction square 90^{o}
- o Place one square on top of the other.
- o Answer: $\dfrac{\text{total number of cross – hatched pieces}}{\text{total number of pieces in one new square}}$

Addition

Introduce addition of fractions as putting pieces together.

T: (says and writes) Celia gives Deion $\frac{1}{5}$ of a brownie and Eva gives Deion $\frac{2}{5}$ of a brownie. If we want to know how many brownies Deion has, how would you draw the fraction squares?

[S divides each square into five pieces, shading in one piece of the first square and two parts of the second (see figure 4).]

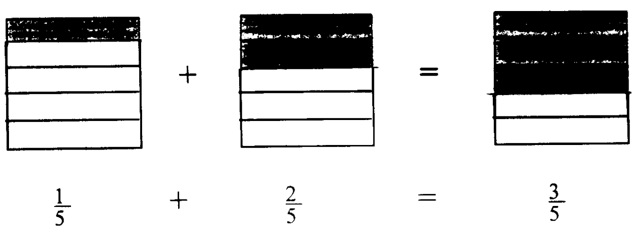


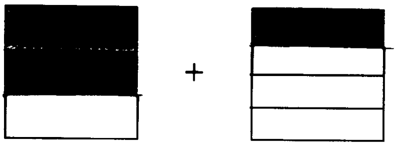
Figure 4. Introducing addition of fractions with common denominators, $\frac{1}{5}+\frac{2}{5}$. The total number of shaded sections (3) divided by the number of sections in one fraction square (5) is the answer, $\frac{3}{5}$.

T: So how many brownie fifths does Deion have altogether?
S: Three. Deion has three-fifths of a brownie.
T: Draw it.
S draws a square, cuts it into five rows, and shades three of them.
T: We can also write down the fractions, one-fifth plus two-fifths equals three-fifths [Writes "$\frac{1}{5}+\frac{2}{5}=\frac{3}{5}$"]. If we didn't have any pictures, how could we find the answer?

S: Add the top numbers and keep the bottom number. One plus two is three. Three-fifths.

After the class has mastered addition problems with common denominators, introduce addition problems with different denominators by putting together food that comes in different-sized pieces.

T: (says and writes) Fiona has two-thirds of a pancake and Glenn has one-fourth of a pancake. If we want to find out how many pancakes they have together, how would we draw the fraction squares? (see figure 5.)



$\frac{2}{3}$ + $\frac{1}{4}$

Figure 5. Addition of fractions with different denominators. A student's initial drawing of the two fraction squares, $\frac{2}{3}$ and $\frac{1}{4}$.

[S draws a square, cuts it into three rows, and shades in two of them (two-thirds). Then, she cuts another square into four columns and shades in one of them (one-fourth).]

T: Since these pancake pieces are different sizes, we have to cut them again to make each piece the same size. Do you remember how we re-cut our food to multiply fractions?

S: Rotate one of them and cut them the way the other one's cut.

T: Show me.

[S cuts the $\frac{2}{3}$ square with four columns ($\frac{2}{3}\times\frac{4}{4}=\frac{8}{12}$) and the $\frac{1}{4}$ square with three rows ($\frac{1}{4}\times\frac{3}{3}=\frac{3}{12}$, see figure 6).]

T: Now we can count them like before. How many do we have?

S: Eleven. Eleven-twelfths of a pancake.

T: Draw it.

[S cuts a square into four columns and three rows, then shades in eleven of the twelve sections.]

T: Now we'll do this without pictures. We started out with two-thirds of a pancake [writes "$\frac{2}{3}$"] and one-fourth of a pancake [writes "$+\frac{1}{4}=$"]. Now how did we cut them?

S: We cut two-thirds with four columns and one-fourth with three rows.

T: Yes, so we cut the fractions that way, too. We cut two-thirds with four cuts, both top and bottom [writes "$\frac{2}{3}\times\frac{4}{4}$"]. So two-thirds times four-fourths is?

S: Eight-twelfths.

T: Yes, eight-twelfths. [Writes "$\frac{8}{12}$"] Now how do we cut one-fourth without pictures?

S: Cut it into three rows, so multiply top and bottom by three and we get three-twelfths [writes "$\frac{1}{4}\times\frac{3}{3}=\frac{3}{12}$"].

T: Yes. Now how do we get our answer of eleven-twelfths?

S: Eight and three are eleven, put it over the twelve, and you get eleven-twelfths.

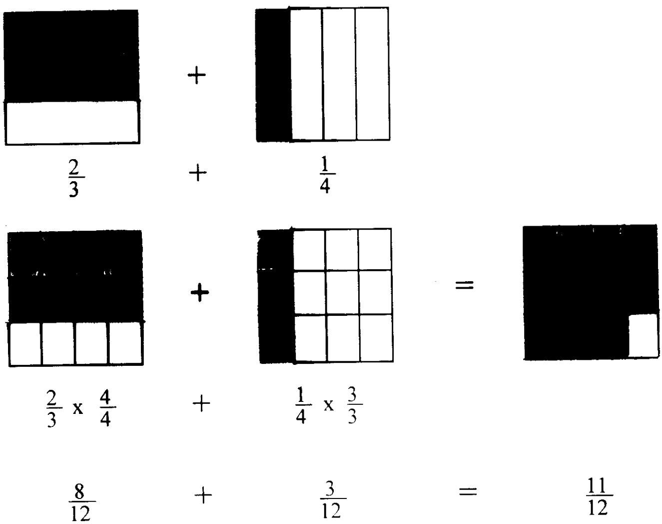


Figure 6. Cut each fraction square the way the other one was cut (4 x 3 and 3 x 4), creating equal sections in each fraction square (12). The number of total shaded regions (11) over the number of sections in one fraction square (12) is the answer, $\frac{11}{12}$.

To add two fractions,

- o Create fraction squares for each fraction, one cut into rows, one into columns
- o Use shading to mark the numerators
- o Cut each fraction square in the same way the other one was cut
- o Answer: $\dfrac{\text{total number of shaded pieces}}{\text{number of pieces in one new square}}$

Subtraction

Introduce subtraction of fractions by taking pieces of food away from the original food portion. Since subtracting fraction squares resembles adding fraction squares in many ways, I will move on to a problem that involves the added complication of borrowing. After our students have mastered subtraction of fractions with different denominators, present the following problem:

T: (says and writes) Sean has broiled $1\frac{1}{2}$ chickens, but his friends only eat $\frac{2}{3}$ of a chicken for dinner. If we want to find out how much chicken is left over, how do we draw fraction squares for the initial amount of chicken and the amount of chicken eaten?

S draws two squares for the original chicken, cuts each one into two rows and shades one full square and one row of the second (see figure 7). Then S draws a square for the chicken eaten, cuts it into three columns, and shades two of them.

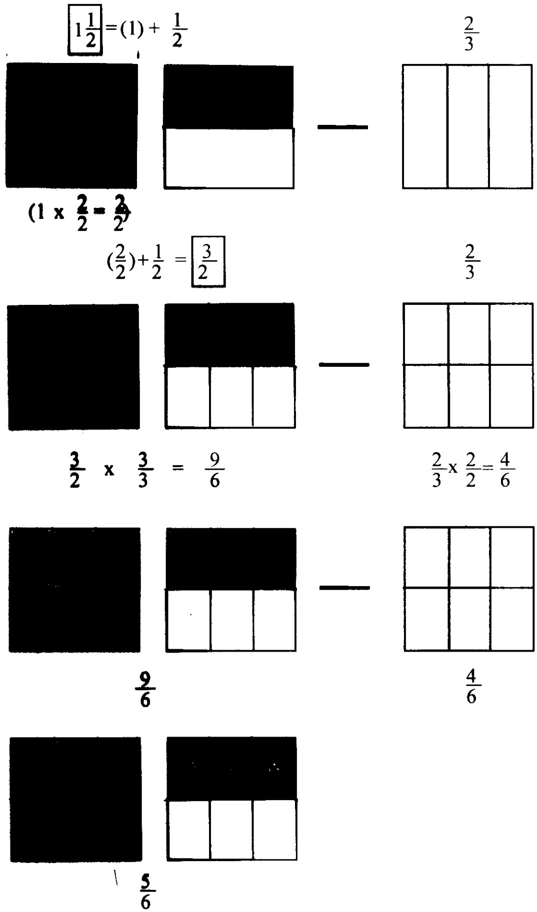


Figure 7. Subtraction of mixed numbers with borrowing. Cut the wholes the same way the fraction square components are cut ($1\frac{1}{2}$ -> $\frac{3}{2}$). Then cut each set of squares the way the other set of fraction squares are cut ($\frac{3}{2}\times\frac{3}{3}=\frac{9}{6}$ and $\frac{2}{3}\times\frac{2}{2}=\frac{4}{6}$). Cross off the eaten chicken pieces (4) from the original chicken pieces (9). The number of remaining original chicken pieces (5) divided by the number of pieces in one chicken square (6) is the answer, $\frac{5}{6}$.

T: Now, how do we re-cut them so that all the chicken pieces are the same size?

S: Cut each one the way the other one's cut.

[S cuts the original chicken squares into three columns and the eaten chicken square into two rows.]

T: How do you figure out how much chicken is left?

S: Take away the eaten chicken from the original chicken. Nine minus four is five. Five-sixths.

T: Yes, five-sixths of the chicken is left over. Now how do we do this without pictures? [Writes "$1\frac{1}{2}-\frac{2}{3}=\frac{5}{6}$"]

S: Cut the whole numbers into pieces like the fraction part. So multiply each whole number by the bottom number of the fraction. So one is cut into two and becomes two halves [writes $1\times\frac{2}{2}=\frac{2}{2}$]. Two halves plus one-half is three-halves [writes $\frac{2}{2}+\frac{1}{2}=\frac{3}{2}$]. Then we cut them the way the other one's cut. So three-halves times three over three is nine-sixths [writes "$\frac{3}{2}\times\frac{3}{3}=\frac{9}{6}$"] and two-thirds times two over two is four-sixths [writes "$\frac{2}{3}\times\frac{2}{2}=\frac{4}{6}$"]. Nine minus four is five, so five-sixths [writes "$\frac{9}{6}-\frac{4}{6}=\frac{5}{6}$"].

To subtract fractions $A\frac{B}{C}$ (original) - $D\frac{E}{F}$ (eaten)

- Create fraction squares for each fraction, one cut into rows, one into columns
- Cut the whole components into pieces (A by C rows & D by F columns)
- Cut each set of fraction squares in the same way the other set was cut
- Cross out the eaten pieces from the original pieces
- Answer: $\dfrac{\text{number of original pieces not crossed out}}{\text{number of pieces in each new square}}$

Division

Introduce division as moving food from old containers to new containers with the following warm-up problem (see figure 8):

> Kai has cooked pizza in a casserole dish and wants to store it in either small, medium, or large plastic containers for a picnic this afternoon. The small container is half the size of the casserole dish, the medium container is the same size as the dish, and the large container is twice as big as the dish. How much of each container will the pizza fill? If Kai had cooked three pizzas, how much of each container would be filled?

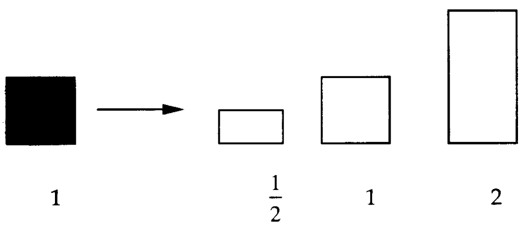


Figure 8. Dividing fractions as moving food (pizza) from the old containers (casserole dish) to new containers (storage container). Since large containers hold twice as much as the casserole dish, the pizza only fills $\frac{1}{2}$ of it. On the other hand, the pizza fills two of the small half-sized containers. Three pizzas fill either 3 medium containers, 1 1/2 large ones or 6 small ones.

After familiarizing your students with this situation, ask the following problem.

T: Kai's family was hungry, so they ate part of the pizza, leaving behind three-fourths of it. If Kai stores the left over in small containers, how should we draw the pizza and small container fraction squares to find out how much of the small containers will be filled? (See figure 9.)

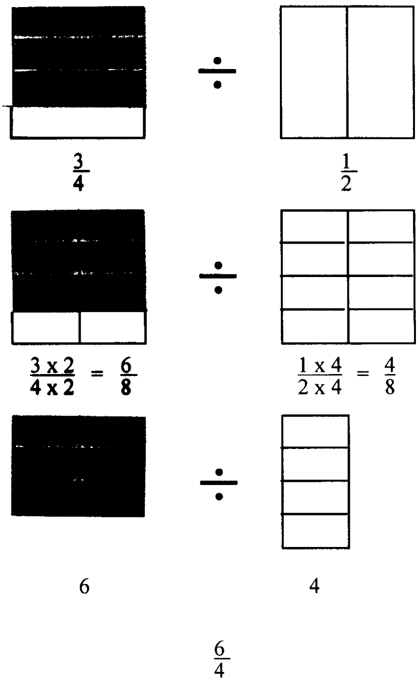


Figure 9. Division of fractions as moving food (pizza) from an old container to a new small container. After drawing fraction squares for the pizza and the new container, cut each fraction square the way the other one is cut to create same-sized sections. Since we are only concerned with putting darkly-shaded pizza pieces into the lightly-shaded new container slots, discard the unshaded sections from the old container. There are six pizza sections to fill four new container slots so the pizza fills $\frac{6}{4}$ small containers.

[S draws two squares, cutting one into four rows and the other into two columns. Then, S shades three rows of the first one (three-fourths) and one column of the second one (one-half).]

T: Since the pizza and the container are different sizes, how should we re-cut them so that all the pieces are the same size?

S: Cut the pizza into halves [draws two columns] and the container into fourths [draws four rows].

T: The blank parts were part of the old container which helped us make all the pizza pieces and container slots the same size. Now, we can ignore the blank parts because we're not using the old container anymore. So, how many pizza pieces do we have?

S: Six.

T: And how many pizza pieces fit into the small container?

S: Four. So one and two-fourths small containers [writes "$1\frac{2}{4}$"]

T: Yes, one and two-fourths small containers or six fourths small containers [writes "$\frac{6}{4}$"]. We can fill up the four slots in the small container once with four of the six pizza pieces and the remaining two pizza pieces go into another container, so one and two-fourth containers or six-fourths containers. How would we do this without pictures? [writes "$\frac{3}{4}\div\frac{1}{2}=\frac{6}{4}$"]

S: Cut three-fourths by two and cut one-half by four. Three-fourths times two over two is six-eighths [writes "$\frac{3}{4}\times\frac{2}{2}=\frac{6}{8}$"] and one-half times four over four is four-eighths [writes "$\frac{1}{2}\times\frac{4}{4}=\frac{4}{8}$"] I guess we throw away the eight's to get six-fourths?

T: Yes. We use the old container to help us cut the pizza and the new container into pieces of the same size. Then, we can ignore the blank pieces because we're not using the old container anymore.

Continue to discuss division ($\frac{A}{B}\div\frac{C}{D}$) as moving objects from old containers ($\frac{A}{B}$) into new ones ($\frac{C}{D}$). (Note that the standard container size is 1.) As our students become familiar with this procedure for dividing fractions, they may omit the denominator multiplication ($D\times B$ and $B\times D$) and simply cross-multiply up ($(A\times D)\div(B\times C)$), yielding $\frac{AD}{BC}$. Cross-multiplying up is equivalent to the standard <u>invert and multiply</u> method. ($\frac{A}{B}\div\frac{C}{D}=\frac{A}{B}\times\frac{D}{C}=\frac{AD}{BC}$).

To divide fractions, $\frac{A}{B} \div \frac{C}{D}$,

- Create fraction squares for each fraction, one cut into rows, one into columns
- Use dark shading to mark the numerator (A) in the $\frac{A}{B}$ fraction square
- Use light shading to mark the numerator (C) in the $\frac{C}{D}$ fraction square
- Cut each fraction square in the same way the other one was cut
- Answer: $\frac{\text{number of dark-shaded sections}}{\text{number of light-shaded sections}}$

You can also use this activity to explain $0 \div \frac{1}{2}$ and $\frac{1}{3} \div 0$. If there is no pizza and the container has some space, then no containers are filled ($0 \div \frac{1}{2} = 0$). On the other hand if there is some pizza and the container has no space, then the pizza fills an infinite number of containers ($\frac{1}{3} \div 0 = \infty$).

Conclusion

These activities tie intuitive understanding to generally applicable algorithms through metaphorical reasoning. Furthermore, the problems can also help students understand arithmetic computations with whole numbers. By choosing situations familiar to students and precisely coordinating each computational step with an analogous real-world action, we can help students understand mathematics intuitively, compute efficiently and build a system of mathematical relationships.