Proof: Why bother?

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At the end of a lecture, a student asked, "Why do we need to learn proofs?"

"What is proof? Why do we need it?" These questions have been discussed on different occasions. For example, Plumpton et al. (1984) say,

> The mathematician has an intuitive, instinctive feeling that some proposition may be true. The essence of proof is to establish whether the result is, indeed, true or whether he has been deceived by such a feeling. .... Fundamentally, mathematical proof is based on logical argument, that is, to establish from a hypothesis 'p is true' a conclusion 'q is true'. (p.1).

The importance of proof in mathematics is evident but why students still raise the question. It may be likely that proof in the curriculum is presented as "an obituary of mathematics" which contains the important facts but not the living sensations that the mathematician felt (Austin, 1991). To capture a mathematician's living sensation may sound ambitious. However, effort should at least be made to help students appreciate the need for a proof and how proof helps them understand the heart of the matter. In what follows, I would like to demonstrate how an ordinary exercise in A-level mathematics can be extended to provide an arena for further learning of the nature of mathematics.

The exercise(*)

To show that there are an infinite number of prime numbers.

A proof by contradiction:

Suppose that there are finitely many primes, say $n_1, n_2, n_3, \ldots. n_k$.

Now consider the number $n_1 \times n_2 \times n_3 \times \ldots \times n_k + 1$.

None of the existing prime numbers, $n_1, n_2, n_3, \ldots. n_k$, *is a factor of* $n_1 \times n_2 \times n_3 \times \ldots \times n_k + 1$. *Thus* $n_1 \times n_2 \times n_3 \times \ldots \times n_k + 1$ ***must also be a prime.*** *This is absurd. Therefore, the initial assumption, that there are finitely many primes, must be wrong.*

Q.E.D.

Referring to the above argument, the following question can be posed:

Is $n_1 \times n_2 \times n_3 \times \ldots \times n_k + 1$ **always a prime?**

After trialling of the first few cases, the guess appears to be more plausible. Nevertheless, verifications cannot be sufficient (see Table 1).

Table 1

The prime numbers	$n_1 \times n_2 \times n_3 \times \ldots \times n_k + 1$	Is $n_1 \times n_2 \times n_3 \times \ldots \times n_k + 1$ a prime?
2	2+1=3	Yes
2, 3	2×3+1=7	Yes
2, 3, 5	2×3×5+1=31	Yes
2. 3, 5, 7	2×3×5×7+1=211	Yes
2, 3, 5, 7, 11	2×3×5×7×11+1=2311	?
2, 3, 5, 7, 11, 13	2×3×5×7×11×13+1=30031	?

The next hurdle will probably be how to check whether a large number "n" (such as 2311, 30031) is a prime. After investigating the factorization of numbers, students may find that they need only to test the divisibility with prime numbers not greater than $\sqrt{n}$. This result will probably resume their interest in Table 1 and they will be happy to find "30031=59×509."

Discussion needs not stop at this stage. It may be proceeded to enhance students' understanding in the following directions:

- How does the hypothesis, "Is $n_1 \times n_2 \times n_3 \times \ldots \times n_k + 1$ always a prime?", arise?
- If $n_1 \times n_2 \times n_3 \times \ldots \times n_k + 1$ does not always give a prime, will the proof still be valid? Why?
- What results have we proved?
- What techniques of proof have been used?
- How observations help in making conjectures or formulating a proof?
- When do we need a proof?

I hope that the above can illustrate that, besides producing a deductive argument to convince, proof serves other purposes in classroom teaching. As Davis and Hersh (1980) suggest,

> Proof, in its best instances, increases understanding by revealing the heart of the matter. Proof suggests new mathematics. The novice who studies proofs gets closer to the creation of new mathematics. (p.151)

References:

Austin, K. (1991). See the butterfly in flight. ***Theta,*** ***5***(2).

Davis, P. J., & Hersh, R. (1981). ***The Mathematical Experience.*** England: Penguin Books Ltd.

Plumpton, C., Shipton, E., & Perry, R.L. (1984). ***Proof.*** London: Macmillan Education Ltd.

(*) The writer used the problem while teaching in a secondary school many years ago. The proof in fact originated from Euclid (Book IX, Proposition 20).