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Creating Learning Opportunities for Mathematical Problem Posing and Using Diagrams in Connection with Pythagoras' Theorem

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Introduction: Initiated by Concerns with Solving (Word) Problems

One of the major goals in secondary school mathematics education is to encourage students to be good problem solvers. Problems which are at the heart of problem solving have been identified by Stanic & Kilpatrick (1989) to serve several major themes of mathematics education. Problems can be employed as vehicles to justify the values of teaching mathematics. They can also provide motivation for learning mathematics and to give students a context to develop new skills. Definitely, one of the major purposes of having word problems at the end of a learning unit is to provide opportunities for students to apply the concepts and skills learnt in that unit. To achieve the goal of developing students' problem solving skills, teachers should try to have problem solving as one of the main mathematical activities and offer students more rich problems in order to provide the students with a challenging and exciting learning environment. Unfortunately, a huge number of word problems, referred as "ready-to-wear" problems by Leung (1996), are always designed in a way that they only enable students to concentrate on the task of choosing a "correct" mathematical operation in order to get a "correct" solution. This tells the general situation with most of our school mathematics problems. Foong & Koay (1997) noted the mechanical responses and "stereotyped thinking" of students who are tackling word problems. When a student reads a word problem, he/she tends “to disregard the actual situation described and instead, they go straight into exploring the possible combinations of numbers to infer the needed mathematical operations” (Foong & Koay, 1997, p. 73). They (*ibid*, p. 74) continued to report, “[M]any studies have shown that the use of standard problems in the classroom hinders pupils’ reasoning and problem-solving ability as it encourages pupils to exclude real-world knowledge and realistic considerations from the solution process.” As a result, more and more students only “see” problem solving as a process of working out a solution from a formula through a series of standard steps.

Admittedly, one of the important purposes of using problems is to provide opportunities of practices, which reflects the general situation with most of the word problems found in our school mathematics textbooks. However, if the generic skill of problem solving is to be developed properly, understanding the problem is of very importance. In order to shift the attention of learning and teaching back to opportunities of comprehending a problem situation and its description by words in the first place, an alternative approach to teaching Pythagoras’ Theorem was devised and tried out in a few Secondary 2 classrooms. We will briefly introduce the approach and the learning tasks involved therein. But, due to limited scope of this paper, we will focus only on one of the tasks, namely the one that prompts the students to pose a real-life problem that is related to right-angled triangles. Inevitably, preparatory skills, e.g. another focus concern with the use of diagrams, and the general readiness to consider such problem situations have been built into the learning tasks previously undertaken in the same lesson. This problem-posing task comes towards the end of the lesson concerned. More details of the teaching approach and the other two learning tasks have been reported in an earlier paper (Yeung, Wong, Cheung & Au, 2019).

Literature Review: Focus on Problem Posing

In order to go beyond the “typical” word problems that induce “stereotyped thinking”, many mathematics education reform documents have called for a change from asking students to solve problems, to pose problems (NCTM, 1989; NCTM, 1991; NCTM, 2000). *The Principles and Standards for School*

Mathematics (NCTM, 2000) suggested that the school curriculum should provide students with opportunities to formulate interesting problems based on a wide variety of situations (p. 258). The *National Statement on Mathematics for Australian Schools* (Australian Education Council & Curriculum Corporation, 1991) also expressed a strong support for students engaging in extended mathematical activities which should engage problem posing. Curriculums of China and England treat problem posing as a means to achieve other important curricular goals such as problem solving and reasoning (Silver, 2013).

Many educators have noticed the important role of problem posing in mathematics education many years ago. Problem posing has been viewed by Polya (1954), Brown & Walter (1983) and Dillon (1988) as an inseparable part of problem solving since mid-twentieth century (Lavy & Shriki, 2007). New problems exist in every moment of mathematical thinking. When mathematicians meet, they share how they find problems, pose new ones, and reformulate novel problems from known problems (Leung, 1996). Brown & Walter (1993) as well as English (1996) also found that providing students with opportunities to pose their own problems could foster more diverse and flexible thinking, enhance students' problem solving skills, broaden their perception of mathematics and enrich and consolidate basic concepts. Cunningham (2004) agreed that problem posing enhanced students' reasoning and reflection. Besides the values of problem posing in mathematics thinking, Lavy & Shriki (2007) suggested that problem posing activities gave students the feeling of ownership in their learning. When students, rather than the teacher, formulate new problems, it fosters the sense of ownership that students need to take for constructing their own knowledge. This ownership may result in a high level of engagement in mathematics education and enthusiasm towards the learning. If we agree that problem posing is conducive to learning mathematics in various ways (Brown & Walter, 1993), undoubtedly should problem posing be included in students' mathematical learning experience.

<u>What is Problem Posing?</u>

Kar, Özdemir, İpek & Albayrak (2010) have noticed the various definitions of problem posing by different mathematics educators. According to Leung (1993), problem posing is the new organization of a given problem. English (1996) defines problem posing as producing new problems and restructuring of a current problem while NCTM (1991) refers problem posing to the formation of a new problem from a given situation or experience. Silver (1994) summarises different perspectives and refers problem posing both to the generation of new problems from a mathematical context and to the reformulation of a given problem during the process of solving it. According to Silver, when problem posing refers to problem reformulation, it usually occurs within the process of solving a complex problem where a solver needs to restate or recreate a given problem in order to make the original problem more accessible to its solution. As Duncker (1945) said, problem solving is a process involving establishing a series of successively reformulated problems of an initial problem that incorporate relationships between the given information and the desired goal, and into which new information is added as sub-goals are satisfied. From this perspective, one of the most important contributions of problem posing is the provision of posing sub-problems for overcoming difficulties faced by the problem solver during problem solving (Kar, Özdemir, İpek & Albayrak, 2010). Problem posing can also occur at the "Looking Back" phase of problem solving suggested by Gonzales (1998) who describes problem posing as the fifth phase of Polya's problem solving. After solving a particular problem, one might examine the conditions of the problem again in order to generate new insights. This kind of questioning helps to generalize and extend the original problems through posing follow-up questions (Brown &Walter, 1993; English, 1997).

However, the term "problem posing" used in contemporary mathematics education reform documents such as NCTM refers to a somewhat different kind of activity, in which problem posing itself is the focus of attention. From this perspective, problem posing is an information source in itself in terms of detection of comprehension level of students in mathematical operations, problem solving skills and attitudes towards mathematics (Kar, Özdemir, İpek &

Albayrak, 2010). The goal of problem posing is no longer to help to solve a given problem or generalize it but create a new problem from a situation or experience. Such problem posing occurs prior to any problem solving and is not dependent on the problem solving process anymore, as would be the case if problems were generated from a contrived or naturalistic situation. This type of problem generation is also referred as problem formulation sometimes, but the process being described here is different from the reformulation that occurs within complex problem solving itself (Leung & Silver, 1997). The term mathematical problem posing used in our try-out refers to the generation of new problems (Silver, 1994) where the problem poser is a provider of information (Simon, 1973). Problem posing is considered as real only when the problem has not been solved by anyone before or the solution unknown to at least the one who formulates it. Leung (1994) discussed several characteristics of this kind of problem posing. Firstly, it is idiosyncratic which means that when one considers some given information and poses a problem, one is trying the various given information with a goal. Secondly, the act of problem posing involves plausible reasoning. Thirdly, a problem posed but not yet solved can be insufficiently specified or impossible.

How to Design Problem Posing Tasks?

Depending on mathematics content, students' levels, learning outcomes and mathematical thinking types, problem posing tasks are classified as free, semi-structured or structured tasks (Abu-Elwan, 1999). Free problem posing tasks are tasks in which students are free to make up any problem without restriction to mathematical content or context while structured problem posing activities are those in which students are invited to formulate new problems from already solved problems by varying the conditions or goals of given problems (Brown & Walter, 1983). For semi-structured problem posing tasks, students will be provided with open-ended situations in which they may construct a problem by using knowledge, skills, concepts and relationships from their previous mathematical experiences. Constructing a word problem in story format is one of the semi-structured activities in mathematics classroom. A word problem in story format is a story problem which incorporates real-life problems and applications

(Ahmad, Tarmize, & Nawawi, 2010). It generally follows a three-component compositional structure: (i) a "set-up" component used to establish the characters and location of the putative story; (ii) an "information" component which gives information needed to solve the problem; and (iii) a question (Gerofsky, 1996).

However, as students may not be able to correctly express in words the relations of the relevant information, a diagram may help to provide concrete evidence of how students conceptualize what they think (Silver, 1996). According to Diezmann and English (2001), a diagram is a visual representation that displays information in a spatial layout. Pape & Tchoshanov (2001), cited by van Garderen and her colleagues (2012), have also pointed out that a diagram is not a static end product. It is both a process – the act of creating or expressing a mathematical relationship or concept – and a product – the objects or representation itself. Brown and Presmeg (1993) found that students with a stronger schematic understanding of mathematics typically generated images more schematic in nature. As a result, it is agreed that representing a posed problem by a diagram helps to reveal how students design a problem, present their result as well as explain their actions (Pape & Tchoshanov, 2001).

Methodology

Background of the Study

Pythagoras' Theorem has attracted voluminous research in mathematics and mathematics education, not least because it has a long history of thousands of years in human civilization. However, educational studies on this topic often focus on students' discovery of the relationship among the three sides of a right-angled triangle as well as the role of its proofs in mathematics classroom. Despite the general belief that word problems on this topic, like those on other curricular topics, show a range of applications of the theorem and, more importantly, provide exercise opportunities for students, the difficulties with word problems are well recognized as mentioned above in this paper. Instead of solving word problems as exercise and application after learning the theorem, this study tries to get students pose word problems related to right-angled triangles before the introduction of the theorem.

There were 122 Secondary 2 students with various mathematics ability from a girl school participating in the try-out. Although most students were quite good in languages, some students were poor in both mathematics and English. Students were all well-trained in rote mechanical skills and preferred to do routine problems during lessons. They were afraid of attempting “new” types of questions and tended to give up if there were no methods readily available.

Research Questions

With the general concern with the pedagogical challenge due to word problems, one of the aims of the trial lesson is to examine the feasibility of a problem posing activity related to right-angled triangles and address the following questions:

1. How do students relate their real-life experiences to the problems they posed?
2. How do students connect different mathematical objects including the numerical data and unknowns in a problem when posing a problem?
3. How do students represent the problems they posed by diagrams?

Design of the Problem Posing Activity

As mentioned in the background outlined above, this study tried to shift the focuses of learning and teaching away from solving word problems as a routine exercise so that the problem posing activity was conducted before learning the topic Pythagoras’ Theorem. In groups of two to four, students worked on three activities. The first two activities were used as stimuli in the problem posing activity in order to prepare the students better for the main activity, the activity three. First, students were required to read four problems and match them with some given diagrams (Appendices 1a and 1b). This activity helps to create an opportunity for students to focus on the textual content of the word problems and encourages them to make sense of the words by means of a diagram. In the second activity, students were required to draw diagrams to represent the situation of another problem (Appendix 2). The last activity which we are going to analyze in this paper is designed to invite students to pose a problem related to right-angled triangles in a story format and draw a diagram representing the corresponding problem (Appendix 3). The products which were going to be analyzed in this activity were the problems themselves. In order to raise their motivation, teachers told the students that some students' work would be chosen as part of the problems to be solved in the last sub-topic "The Applications of Pythagoras' Theorem" (Appendix 4).

As these three activities were conducted before the introduction of Pythagoras' Theorem, the thinking process possibly going on in these activities were, according to the design, expected to focus only on the comprehension of the word problems and sense-making of the real-life situations but not any explicit need for calculation and formulas associated with Pythagoras' Theorem.

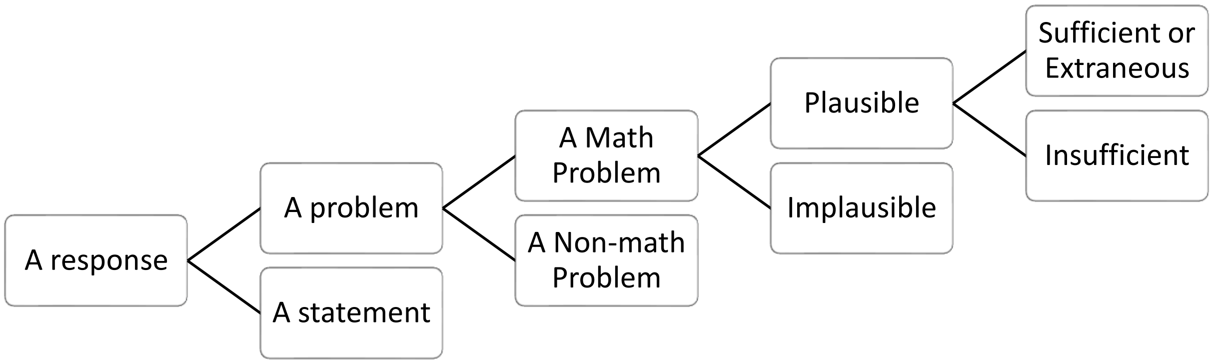
Method of Analysis

For this paper, focus will be put on the responses produced by the students to the problem-posing tasks. Besides general observations in the classroom and after the lesson, we also try to conduct qualitative analysis of the responses submitted by students. As we believe that students who have various backgrounds and different abilities may possess different potentials in thinking patterns, imagination, fantasy and performance, it is reasonable to posit that the students have different levels of mathematical knowledge and thinking. It is agreed that, in order to generate a reasonable mathematical problem related to a given situation, one must be aware of facts and relations embedded in the situation; be able to mathematize the situation and be able to present one's mathematized situation in the form of a problem (Leung & Silver, 1997).

In fact, some previous research pointed to differences in problem posing between students with high mathematics ability and those with low mathematics ability. Both Ellerton (1986) and Krutetskii (1976) reported that more capable students appeared to be more thoughtful in their problem posing as they could see those problems that naturally followed from the given information. Silver and Cai (2005) have identified that there are several facets embedded in a problem including problem difficulty, linguistic complexity and mathematical complexity.

Therefore, a problem can be examined from various perspectives. For the purpose of evaluating students' mathematical understanding and cognitive processes, the students' responses to the problem posing task should be assessed based mainly on the mathematical complexity along two dimensions: quantity and quality. Quantity refers to the number of correct responses generated from the problem posing task. Counting the number of correct responses may be deemed by many as a trivial way of evaluating students' response to generative activities such as problem posing. However, the fluent generation of responses can potentially inform the teacher about students' characteristics.

With respect to the quality, posed problems can be analyzed based on the analytic scheme proposed by Silver and Cai (1996). The scheme classifies responses of students according to four criteria: response type, problem type, feasibility of initial statement and data required in solving the problem. This can be conducted in a four-step process. First, responses are classified into problems or statements. Non-problems will be sieved out before focusing on those that are mathematical problems. Next, a problem is classified either as a mathematical problem or as a non-mathematical problem. Non-mathematical problems are problems which are not necessarily solved by mathematics. Each response classified as being a mathematical problem will be further classified as either plausible or implausible. A response is judged to be plausible if the initial state of the posed problem appears to be feasible and no discrepant information can be found while an implausible problem consists of invalid pre-suppositions which makes the initial state of problem impossible to exist. The final step in the classification process occurs when all plausible responses are further analyzed with respect to the sufficiency of the information provided for solution of the posed problem. Insufficient problems are different from implausible as an insufficient problem can be solved if missing information is added but no answer can be found from an implausible problem even when more information is supplied. A problem is judged as sufficient if it is solvable by information found in the sentences itself. Problems with extraneous information are also considered to have sufficient information as long as they can be solved with some subset of that information. The classification process is shown in the following figure:



Analysis and Results

General Observations of the Problem Posing Activity

Since students were not familiar with this new kind of learning and teaching activity, they found it difficult to make up their own problem. Students felt puzzled and raised various queries in the classroom or after the lesson. Some able students were concerned with the possible appearance of the problem-posing task in the examination. However, most students were excited to know that the problems they posed would be possibly selected to be part of the mathematics exercise when coming to the sub-topic “The Applications of Pythagoras’ Theorem” at a later stage. Below are some examples of questions raised by some students during the activity:

- What kind of problems do you expect me to pose?
- Can you give me some examples?
- Do I need to solve the problem?

The first two listed above reflect very well the usual uncertainty that students have about the requirements put forward by their teacher. The last one typically indicates the doubts that the students might have about problem posing, because they had been very much used to solving problems in mathematics lessons. It should have been absolutely strange to end up with a problem and take it – an unsolved problem – as the product of their work.

Analysis of the Problems Posed by the Students

After the lesson, 82 worksheets have been collected. The next few pages report the results of our analysis of the students' responses to the problem-posing task. For illustration purpose, examples of the problems posed by the students are reproduced here in italics. To give the readers a good sense of what some students (perhaps above-average ones in general) are thinking, we keep the original sentence structure and choice of words, which sometimes appear as clumsy and awkward. But, to grant the ease of reading, minor spelling mistakes and obvious grammatical errors are corrected.

Apparently, in such a problem-posing task, there can hardly be any "correct" responses clearly distinguishable from "incorrect" ones. But it can be found that 67 (81.7%) of the 82 responses are clearly related to right-angled triangles and the other 15 responses (18.3%) are not. The former type is considered as relevant responses whereas the latter irrelevant. ("Irrelevant" as they are considered is only because of our current research purpose and ease of reporting. Pedagogically speaking, for the benefit of the students in the process of learning, they are not irrelevant.) It is obvious to notice that most of the stories show contexts most likely coming from the students' real-life experiences. These experiences include "*A library book leans vertically against a shelf.*", "*There is a slide in a park.*", "*The length of Aimee's chair back is 40 cm long.*", etc. Perhaps due to the historic attack of the super typhoon Mangkhut a few months before the lesson, typhoon (even that with the specific name) is also a favourite setting found among their stories. For example, "*A tree is x cm tall. After the typhoon Mangkut, the tree trunk broke apart.*" Expectedly, their teachers are always the main characters in many stories. For example, "*Mr. Aaron Wong is trying to use a rope to save Mr. Peter Au.*" Sometimes, the background of a story is likely to be related with some unique personal experience. Here are some examples:

- *A rhododendron is 0.5 m tall. Its upper part is rotten and decayed to touch the ground.* (by a student who has taken geography as her elective subject)
- *A man is sliding down towards a snowy mountain and his friend is waiting for him at the other side of the mountain.* (by a student who has experience in skiing)

- *One day, a man went to play the megazip. He was lifted up by x m and then slided down by 17 m.* (by a student who has once commented that “Megazip” is a game full of fun)
- *An Alpaca which has a long neck was 1.2 m tall*. (by a student to whom alpaca, though an uncommon animal species, is a lovely animal)

Besides the contexts suggested by the story plots, the contents of the story can also be considered and analysed with a mathematical lens, i.e. in terms of the mathematical relevance and reasonableness of the measurements involved. From such a perspective, some interesting cases can be noticed. For example, although many stories suggest simple geometric configurations in which right-angled triangles occur apparently, some stories reveal in the first place other polygons in which right-angled triangles are embedded. One of such stories is, “*Amy is 1.5m tall, John is 1.7m tall and the distance between them is 4m. Find the distance between the heads of the two teens.*” Sometimes the measurement data did not have any units while the units of some other measurement data were not properly included. Examples are: “*Two toy bricks on the top of a box. The triangular brick is 8 long and the length of the box is 12.*” and “*A wolf blew the house built by three pigs. The house was 2m tall and the wolf was 6 feet away from the house.*” There are few cases which are apparently correct in terms of the measurement kind but unrealistic in nature. For example, “*The tree trunk is 20cm tall and the park is 25cm long.*” and “*A sword which is 5m long rests against a vertical wall.*”

Among the 15 responses which are considered as irrelevant, three of them do not seem to suggest any story problem while seven others *do* show some reasonableness but the written texts have been produced as isolated words or phrases such as “*wind blow, the box fall down*” and “*lamp fall because of typhoon*”. These students showed some ideas about the plots of their stories but were unable to complete or articulate them in proper (English) language so as to put together reasonably a relevant situation. There were five students who managed to make some story-like plots which however did not seem to reveal any distinguishable right-angled triangles or the like. For example:

- *The height of the bear is 20cm. The floor 25cm long. How long is x?*
- *Cooky is a cute doll sitting straight on the bed. A child hits it down accidentally and it tilts to the left by 30cm on the bed. The bed is 10cm long. What is the length of the doll?*

The 15 irrelevant responses have been excluded from further analysis of the problem posed (as far as the component of words is concerned). For the 67 relevant responses, qualitative analysis was conducted following the framework described above.

Firstly, our findings indicate that 63 (about 94.0%) of the 67 relevant responses give story problems which are expressed in complete or nearly complete sentences; and they *all* can be considered as mathematical problems which may be solved in a mathematical way. (The two examples in Appendix 4 may help to show the kinds of sentences that the students managed to produce.) The remaining stories fail to provide a problem because they are expressed in the form of statements instead of questions nor anything that asks for a response. Typically, as shown in the example below, students may have forgotten to state the question after writing an elaborate story plot:

> *A mother wants to save her son from a robber. However, her son was stuck on the 15th floor of a building. To enter the building, it needs passwords which mother doesn't know. Mother uses a ladder to enter the 15th floor which is 30m high and she is 10m far from the building.*

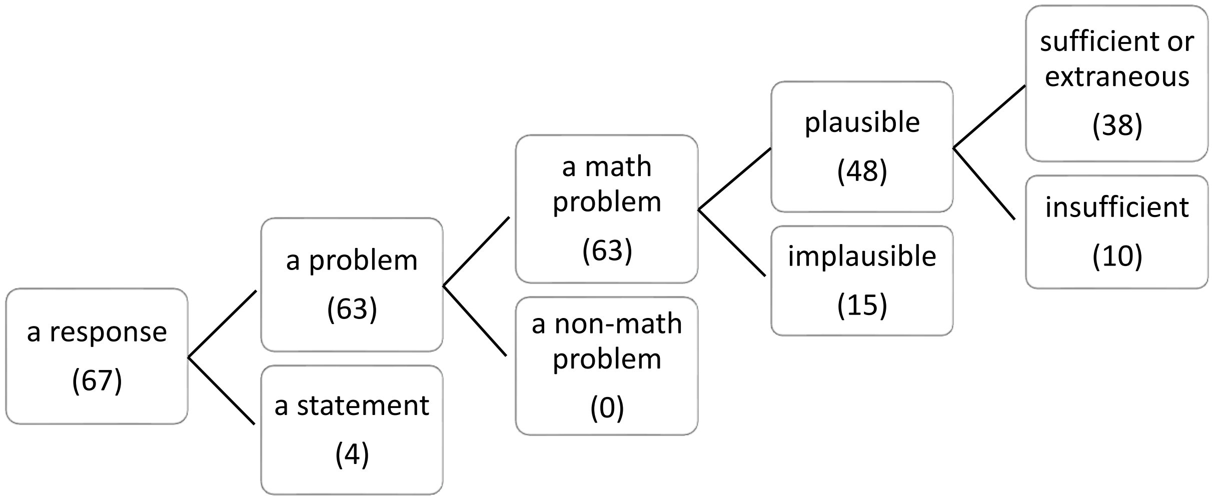
Among all the 63 mathematical problems, 48 (about 76%) are plausible. Problems are implausible mainly due to wrong initial assumptions or conditions. One of the wrong initial conditions, perhaps particularly relevant to the learning of Pythagoras' Theorem, is that the length of hypotenuse is shorter than either of the other two sides of the right-angled triangle. For example,

> *There is a door in 2D classroom which is 2m high. After an earthquake, the door broke. The main part and its broken portion form a triangle. The top of the door touches the ground 3m from the bottom part of the door. What is the length of the bottom of the door left standing?*

Among all the plausible mathematical problems, about 79% (38 of 48) are sufficient. There are more than one group of students who managed to provide their own answers for checking, although the solution is sometimes wrong. Noteworthy is one problem that includes a prescription, "*give the answer correct to 3 significant figures.*" It appears to show an awareness of the accuracy of the answer or an attempt to follow the usual practice of a word problem.

On the other hand, there are problems that can be considered as "insufficient" due to various reasons. First, different words were used to represent the same object such as "Peter and Tom", "toothbrush and chopsticks" and "ruler and spear", which result in ambiguous interpretation of the problem. Second, the unknown x was undefined in the problem. For example, "*A pen straightly stands on the floor and the pen is 13cm. It falls on the floor. What is the value of x?*" Third, there is insufficient information. For example, nothing about the speed is provided together with the time (30s) of sliding down in this suggestion, "*A man spends 30s sliding down from a snowy mountain. Suppose the length of the sliding path is two times the height of the mountain. What is the height of the mountain?*" Lastly, the wordings of the question part are ambiguous and clarification must be sought. "*Country A needs to walk 100 km and climb a 251 km mountain to attack Country B. What is the distance the people need to walk?*" Due to the ambiguous meanings of "*walk 100 km*" and "*climb 251 km*", the question about "*distance*" is not at all clear.

In summary, out of the 82 problems posed by the 82 groups of students involved in this study, 67 problems were analyzed in terms of its quality as a mathematically plausible and sufficient problem. The varieties of problems posed can be summarized in the figure below.



Analysis of the Diagrams of the Posed Problems

As mentioned before, it was not easy for students to put all the relevant information together into a word problem. A diagram may help the students to express their ideas explicitly. Two classification models have been used to analyze the diagrams provided by students. One is the classification scheme proposed by Hegarty & Kozhevnikov (1999) and the other the categorization method adopted by Uesaka & Manalo (2006).

By the classification scheme adopted by Hegarty & Kozhevnikov (1999), diagrams are classified into *pictorial representations* which are primarily drawings of objects, and *schematic representations* which are images representing the spatial relationships among objects described in the problems or combinations of pictorial and schematics representations.

According to Uesaka & Manalo's categorization method (2006), qualities of diagrams will be assessed in terms of both the *structure* and *the information contained in the diagrams*. With respect to the *structure*, a diagram will be placed in the higher Category A if it represents the situation exactly the same as the given word problem. Otherwise, it will be placed in the lower Category B. With respect to the *information contained in the diagram*, diagrams will be classified into 5 types, Categories A to E, according to the amount and the kinds of relevant information in the diagrams. A diagram is placed in the highest Category A if it contains additional inferences drawn from the problem given, but it will be placed in the next, Category B, if it contains all the numbers specified in the problem without evidence of additional inferences. A diagram is placed in the Category C if it contains some of the numbers specified in the problem, but in the Category D if it contains some numbers but all of them are incorrect or unrelated to the problem. At the other end of the scale, the lowest category E includes those diagrams which contain no numbers at all.

Admittedly, the original purpose of Uesaka & Manalo's (2006) categorization aims at studying how students comprehend a given word problem and visualize the problem with the aid of a diagram in the process of problem solving but not how students make use of their diagrams to express the ideas about the problems they want to pose. If we take a look at Mayer's (1985) idea about the process of problem solving, we find that the process of solving a problem is comprised of two major phases: *problem representation* and *problem solution*. Problem representation can be divided into two sub-stages: *problem translation* which relies on linguistic skills needed to comprehend what the problem is saying, and *problem integration* which depends on the ability to mathematically interpret the relationships among the problem parts to form a visual representation. Problem translation is concerned with how a student reads the problem for understanding while problem integration focuses on how the student puts the parts together and comes to *visualize* the problem in a holistic representation. Uesaka & Manalo's (2006) categorization helps to evaluate how students conceptualize what they read through a visual representation in the sub-stage "problem integration". In a similar but not exactly the same manner, the categorization can also help to analyze how a student makes use of a diagram as a means to further explain the problem he/she is posing or to formulate the problem in the more dynamic process going back and forth between words and diagrams. However, it would be appropriate for us to modify Uesaka & Manalo's (2006) categorization method by adapting each category to the current task of problem posing. The modified criteria used for evaluating the quality of diagrams produced by students in problem posing are shown in the table below:

Area of evaluation	Categories	Criteria for placing in the category
Structure of diagram	A	The diagram represents the situation exactly the same as the posed problem or helps to further explain the situation that the student has not described very well in the problem posed.
	B	The diagram does not represent the situation in the problem posed.
Information contained in the diagram	A	The diagram contains additional information that the students has not described very well in the problem posed.
	B	The diagram contains all of the numbers correctly described in the problem posed.
	C	The diagram contains some of the numbers described in the problem posed.
	D	The diagram contains some numbers but all of them are incorrect or unrelated to the problem posed.
	E	The diagram contains no numbers.

Among the 82 diagrams attached to the 82 problems collected, 11 diagrams (13.4%) show pictorial representations while 53 (64.6%) are schematic representations. The remaining 18 diagrams, i.e. about one-fifth of the diagrams collected, show a combination of pictorial and schematic representations produced in this problem posing task. Among those who showed a combined usage, the teachers also observed that some students had drawn pictorial images first and eventually transformed these images into schematic representations. In short, it shows that many students (over 80% of the student groups) participated in the study are ready to use schematic representation, perhaps with the aid of pictorial representation in some ways, in the visualization of the problems they are posing. Some examples of students' representations are shown as follows:

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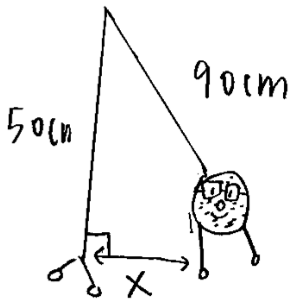
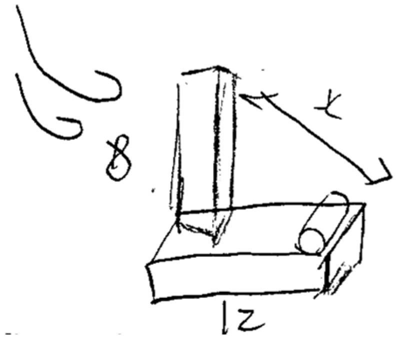
<tr><td>Pictorial Representation</td><td>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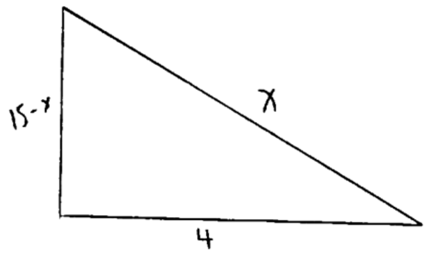
<tr><td>Schematic Representation</td><td>
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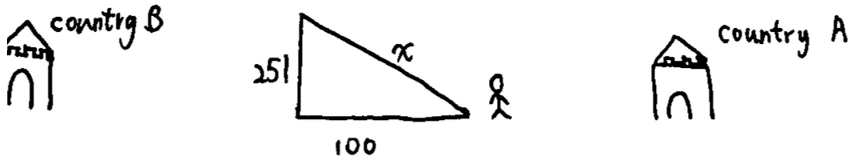
<tr><td>Combination of Pictorial and Schematic Representations</td><td>
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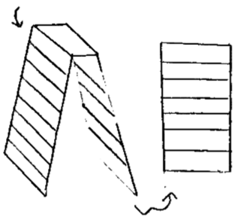
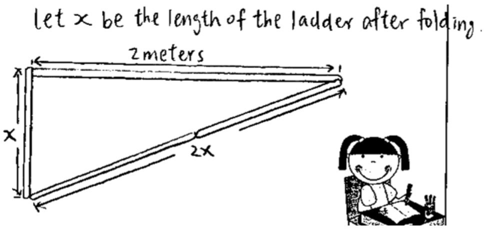
<tr><td>Transformation of Pictorial Representation to Schematic</td><td>

</td></tr>
</table>

We have also noticed that, 10 out of 82 responses include diagrams but do not pose clearly any problem in whatever sense. The students might have some ideas about their story problems and were able to represent the situation by a diagram. However, their language ability was not good enough to express the situation verbally nor to articulate the problem clearly. As far as the relationship between a posed problem and its corresponding diagram is concerned, these 10 diagrams are excluded in the further analysis. The remaining 72 diagrams are classified according to an adaptation of Uesaka & Manalo's (2006) categorization as described above, and the distribution of student diagrams among the categories is shown in the tables below.

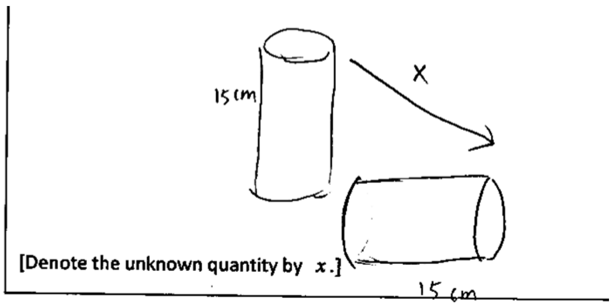
Area of evaluation	Category A	Category B
Structure of diagram	84.7%	15.3%

Area of evaluation	A	B	C	D	E
Information contained in the diagram	19.4%	54.2%	16.7%	8.3%	2.8%

In the previous sub-section with an analysis focusing on the problem posed in words, we have noticed that 81.7% (67 out of 82) of the student groups managed to pose a problem related to right-angled triangles in complete or nearly complete sentences. Now, considering only those who have worked on words in connection with diagrams, we have a slightly higher percentage 84.7% (61 out of 72) student groups who could pose a relevant problem. Although such an increase in proportion of students may not be very significant, it aligns with the general belief that visual representations would help students further explain or facilitate the comprehension of a problem situation. Instead of any definitive evidence of the positive use of diagrams, it encourages us to examine in greater detail the diagrams produced by students in this problem posing activity. With illustration by student examples displayed in the following page, we have tried to elaborate the varied uses of diagrams in posing a mathematical problem.

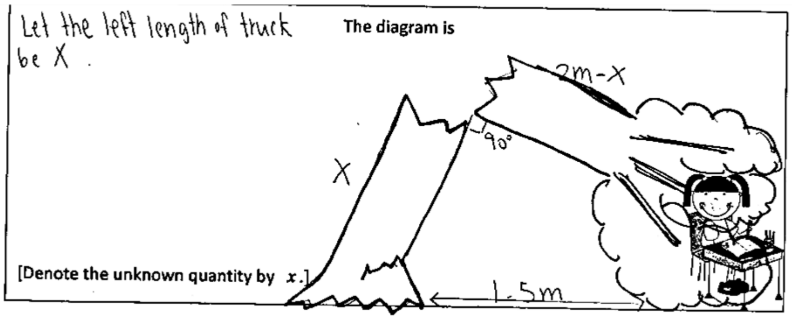
Firstly, a diagram helps to clarify the meaning of the “question part” (e.g. as found in Example I below, what could “difference” mean?). Secondly, a diagram helps to locate the orientation of the right-angled triangle (especially when the situation is not as usual as it is, as shown in Example II below). Thirdly, a diagram helps to decode the descriptions given in words for the relationship among different mathematical as well as real-world objects, when the number of objects come together in a more intricate interrelationship (as shown in Example III below).

Example I



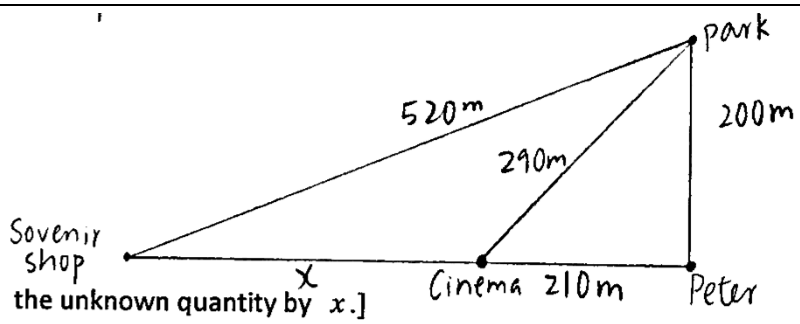
Tom accidentally pulls down his water bottle. The height of the water bottle is 15cm. After the bottle falls down, it is in a horizontal position. The length is the same as the height. Find the difference when the water bottle is standing stood and lying down.

Example II



A typhoon is coming. Unluckily, a tree is blown by the strong wind and has broken off. The original length of the tree is 2m but now there is only a trunk left. ***The broken part and the left part formed a right-angled triangle*** *which the broken part is 1.5m far away from the trunk. What is the length of the trunk that is left?* [bold added]

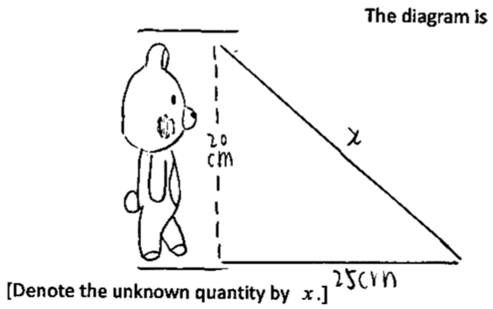
Example III



The road from Peter to a souvenir shop is a straight road. The shortest distance between Peter and a park is 200m. When Peter walks 210m to the cinema to meet his friend Johnny, the distance between him and the park is 290m. If the distance between the shop and the park is 520m, how far does Peter have to walk from the cinema to the souvenir shop?

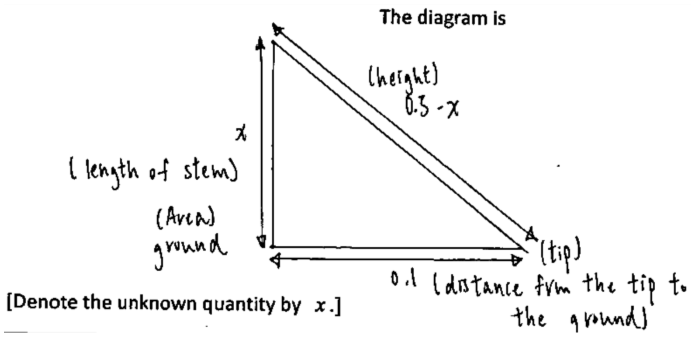
Referring to the information contained in the diagram, most students could construct diagrams according to their own problems while about one-fifth of the diagrams supplied additional information which had not been mentioned in the problem posed. A diagram helps readers to locate the unknown variable x (see Example IV). A diagram also helps the problem poser to infer from the context of the problem being posed as in Example V below, where $(0.5 - x)$ is a measurement (length) that has been inferred from the information given in the description of the situation. This, in turn, helps readers to comprehend the problem because the inferred details make the situation more visible and transparent. However, 20 (about 28%) of the 72 responses pose problems with certain details but they do not associate with correct diagrams. For example, the information marked in the diagram of Example VI does not connect easily with the words.

Example IV



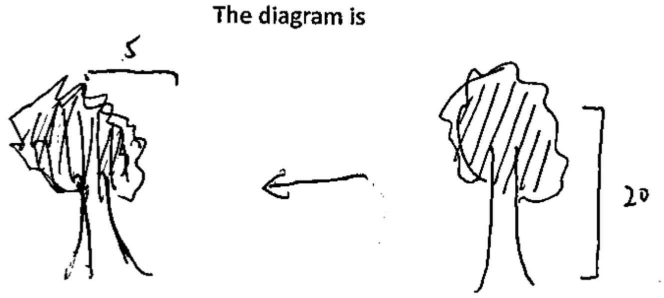
The height of the bear is 20cm. The floor 25cm long. How long is x?

Example V



A Rhododendron is 0.5m tall. The upper part is rotten and decayed. It has broken into two parts near the top and falls down. The lower and its broken part form a triangle. The top of the flower touches the ground 0.1m from the stem. What is the length of the stem left standing?

Example VI



A tree of 20m is standing vertically above the ground. The tip of the tree touches the ground 5m from the stem. What is the length of the stem left?

Teachers' Reflection

Confronted with the problem-posing tasks, students started with some puzzlement. With some prompts and clarification (e.g. drawing connections back to the previous learning tasks in the same lesson), the meaning and the purposes were better communicated. Admittedly, the teachers faced similar difficulties as their students did when conducting this problem-posing activity. As we had very little opportunity to experience problem posing in our teaching before, we tended to think of the usual pedagogical skills, rules and procedures instead of keeping ourselves focused on the aims of the activity itself which is an instrument for developing problem solving and reasoning skills. In our everyday teaching, when students are given opportunities and time to draw diagrams themselves, diagrams, together with their meanings to students and their supportive role in the learning and teaching process, may have been easily overlooked. Not until we looked into the varieties of problem situations and their representations in diagrams produced by students, we did not seem to be able to recognize the range of difficulties that students might have in connecting the details of a problem situation, some of which mathematical whereas some others irrelevant, with an appropriate diagram. Very likely, just as we have failed to help students comprehend problem situations, we have missed the opportunities to help students build confidence in managing unfamiliar learning tasks. Although we rarely use problem posing and we do not possess the required skills, we come to realise that incorporating problem-posing activities in our lessons may enable students to become better problem solvers. We hope that we will have chances to further explore the pedagogical uses of problem-posing tasks and of diagrams as helpful representations of problem situations. Continued attempts would be made on curriculum topics/units other than Pythagoras' Theorem.

Conclusion

Problem solving has been identified as one of the major goals for mathematics education in Hong Kong. Unfortunately, solving "ready-to-wear" problems day by day tells the general situation occurring to most of our classroom mathematics education. On the other hand, it has long been proposed that mathematical problem posing can play a central role in enriching the learning experience in mathematical areas. If our understanding of mathematical activity is to increase our students' capacity in mathematical thinking, the learning task as reported in this paper has given us a wonderful experience to begin our journey in problem posing. Our try-out on Pythagoras' Theorem, a topic which has been very familiar to most teachers, neither focused on introducing Pythagorean relationship nor proofs of the theorem, but took a non-routine approach in order to encourage students to dwell into relevant real-life problem situations through posing a problem related to a right-angled triangle before they actually learn the new Pythagorean formula.

In the activity, we noticed that students loved problems related to real-life situations and tried to connect their daily life experiences with mathematics. It is found that a diagram may help students to express their ideas about story problems especially for those whose language ability is not very good. Diagrams are often promoted as a tool or supportive means for students' understanding of how things come together, be they mathematical entities, measurements, or real-life objects. It is thus found that diagrams support student attempts to work on this problem-posing activity. However, we have also noticed some students' inadequate understanding of mathematical concepts such as relation amongst three sides of a right-angled triangle, speed and distance as well as misuse of units. All these come back to the fundamental issues with the capacity of students in making sense of mathematical concepts and their inter-relationships once they are embedded in real-life situations. This sense-making capacity cannot be much enhanced if learning opportunities continue to be confined to routine practice of plugging-in formulas in typical word problems describing familiar situations. In conclusion, as many have alerted us to the phenomenon of "stereotyped thinking" induced by "standard" word problems, developing new learning and teaching strategies, such as the learning activities focusing on problem posing and use of diagrams, should deserve more pedagogical investigation.

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Appendix 1a

Real-life Problems	Diagram
1. A spear 2 m long rests against a vertical tower. If the foot of the spear is moved 1.2 m away from the tower, how far up the tower does the spear reach? 	
2. A bamboo 10 m tall breaks into two parts near the top. The main part and its broken portion form a triangle. The tip of the bamboo touches the ground 3 m from the stem. What is the length of the stem left standing? 	
3. An erect pole of 5 m is standing vertically about the ground. Suppose the base of the pole is moved out 3 m now. What is the distance that the top of the pole is lowered? 	
4. A reed stands against a wall. If the top of the reed moves down 9 cm, the lower end of the reed slides away 27 cm. How long is the reed? 	

Appendix 1b

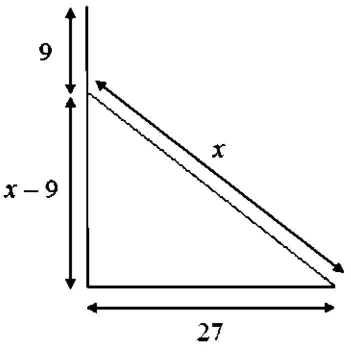


Diagram A

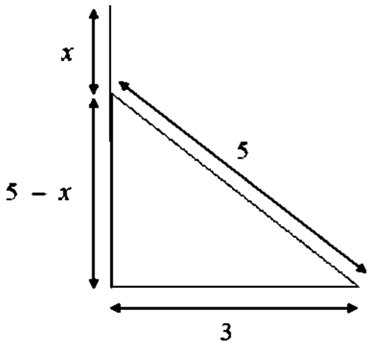


Diagram B

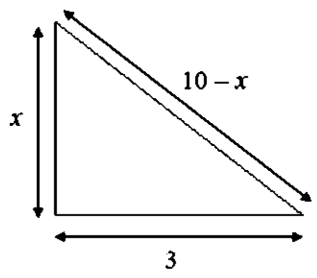


Diagram C

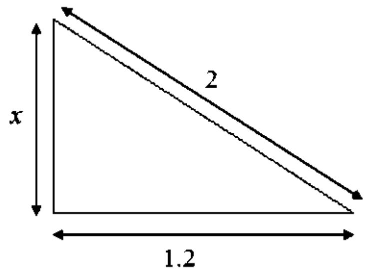


Diagram D

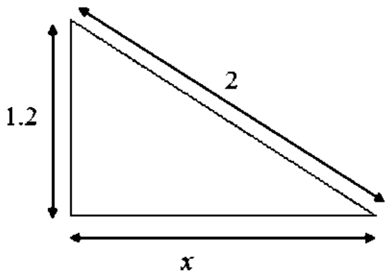


Diagram E

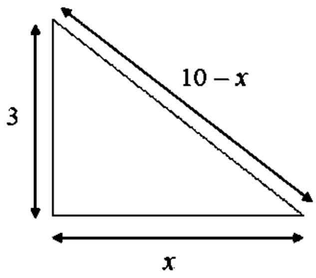


Diagram F

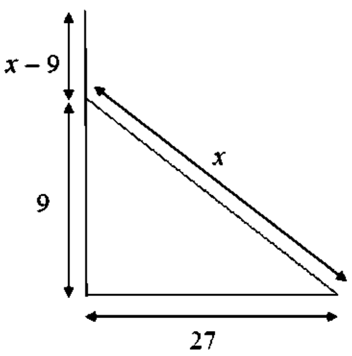


Diagram G

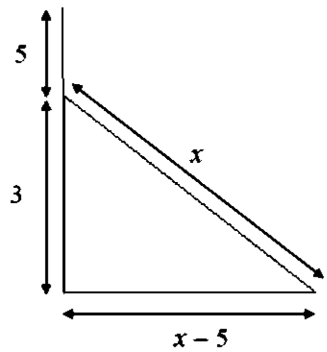
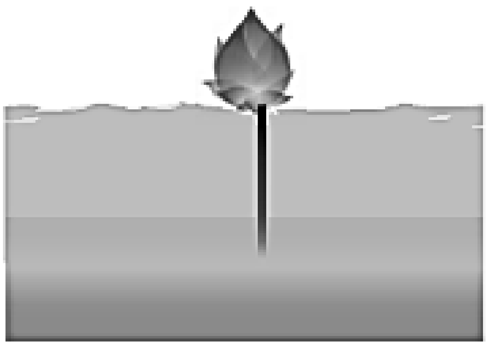


Diagram H

Appendix 2

A Lotus in the Lake (India)

There was a lotus standing in a lake. The tip of lotus was 0.5 feet above the water. The wind blew the lotus and it swung and was submerged at the distance of 2 feet. What is the depth of water?



Let x feet be ____________________________________.

Appendix 3

S2 Mathematics **Story Writing Worksheet**

Class : ___________________ **Name :____**

Create a word problem involving a right-angled triangle.

What is your problem about?

What quantity you want to find out?

What information is needed to find the unknown quantity?

Our Word Problem

____

____

____

____

____

____

____

____

____

____

____

____

The diagram is

[Denote the unknown quantity by x.]

Appendix 4

(Examples of questions posed by students)

S2 Mathematics Chapter 10 Pythagoras' Theorem

Class Activiy: Applications of Pythagoras' Theorem

1. One day, a man went to play the megazip. He was lifted up by x m and then slid down by 17 m. If the horizontal distance that the man moved is 15 m, what is the value of x?

2. Mr. Peter Au and Mr. Aaron Wong leave port O at the same time in different directions. Peter walks 15 km due east and Aaron walks 10 km due north. What is the distance between the two people now?