An Unusual Way of Proving $sin(tan^{-1} 3) = \frac{10}{\sqrt{3}}$ via Solving a Reframed HKCEE Problem

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Introduction

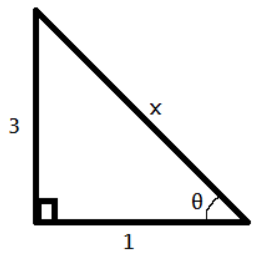
A common teaching practice is to adopt a top down approach in which the teacher starts from displaying a concrete problem and then works out the solution step by step till the end. Apparently, such teaching flow is too linear and the learning outcomes can be predicted in advance easily. In order to make the lesson to be more inspiring and interesting, we propose to start with a seemingly unrelated question and then ask students to solve a repackaged problem leading to some other results which are the main courses to be introduced originally. In this paper, we will describe our teaching idea based on this rationale. More precisely, students will be asked to prove that $sin(tan^{-1} 3) = \frac{10}{\sqrt{3}}$ ---- (*) at the beginning of the lesson. It seems to be a straightforward question testing their knowledge of Pythagoras' Theorem and basic trigonometry only. However, we also tried to reframe a popular HKCEE problem about geometry [1] as a long question such that its solution reveals some other bonus results including a possible solution for the starting question (*). By asking the students to do the reframed problem, they will learn not only the way to solve the original HKCEE problem but also appreciate the beauty of solving problems through different strategies.

Teaching Idea

First of all, teacher asks students to try the following problem:

Question: Show that $sin(tan^{-1} 3) = \frac{10}{\sqrt{3}}$.

If students got stuck, teacher could prompt the students by considering the following diagram:



It is expected that students would be able to give the following solution:

$\tan\theta = \frac{3}{1} = 3 \Rightarrow \theta = \tan^{-1} 3$,

$x^2 = 3^2 + 1^2$ (Pyth. Theorem) ,

$x = \sqrt{10}$.

So, $\sin\theta = \frac{3}{x} = \frac{3}{\sqrt{10}}$

$\Rightarrow \sin(\tan^{-1} 3) = \frac{3}{\sqrt{10}}$.

Next, students were told that the above outcome could actually be resulted in solving a HKCEE modified problem. Teacher will then ask students to try completing the worksheet for the problem depicted below:

(Problem statement of a modified HKCEE multiple choice question)

Refer to the diagram shown below (see Figure 1). *ADB*, *AEC* and *DEF* are straight lines. Given that $AD = DB$ and $DE : EF = 2 : 5$. Let $A = (4, 4)$, $B = (0, 0)$ and $F = (12, 0)$.

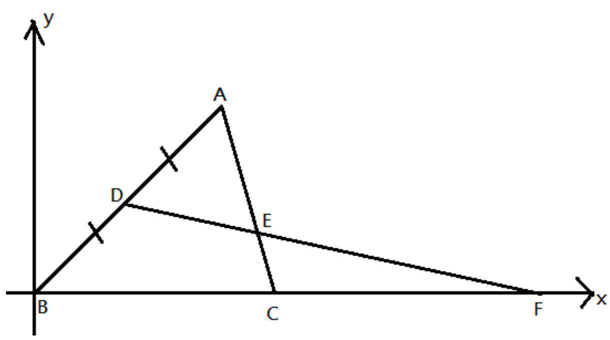


Figure 1

(a) Find the coordinates of D and E.

Solution

$D = \left(\frac{0+4}{2}, \frac{0+4}{2}\right) = (2, 2)$.

By section formula, we have:

$E = \left(\frac{5\times2+2\times12}{2+5}, \frac{5\times2+2\times0}{2+5}\right) = \left(\frac{34}{7}, \frac{10}{7}\right)$

(b) (i) Let $\angle ACB = \theta$. By considering the slope of AC, find $\tan\theta$.

Solution

Slope of $AC =$ Slope of $AE = \frac{4-\frac{10}{7}}{4-\frac{34}{7}} = \tan(180^\circ - \theta) = -\tan\theta$.

Thus $\tan\theta = 3$.

(ii) By considering the y-coordinates of A , E and C , find the ratio of $AE : EC$.

Solution

Let $AE : EC = m : n$. By section formula, we have:

$\frac{10}{7} = \frac{n(4)+m(0)}{m+n}$

$\Rightarrow 10m + 10n = 28n$

$\Rightarrow 10m = 18n$

$\Rightarrow \frac{m}{n} = \frac{18}{10} = \frac{9}{5}$.

i.e. $AE : EC = 9 : 5$.

(iii) Express the length of EC in terms of $\sin\theta$.

Solution

$\sin\theta = \frac{\frac{10}{7}}{EC}$

$\Rightarrow EC = \frac{\frac{10}{7}}{\sin\theta} = \frac{10}{7\sin\theta}$.

(c) By using the results obtained in (b), show that $sin(tan^{-1} 3) = \frac{3}{\sqrt{10}}$.

Solution

By distance formula, $AE = \sqrt{\left(4 - \frac{34}{7}\right)^2 + \left(4 - \frac{10}{7}\right)^2} = \sqrt{\frac{360}{49}} = \frac{6}{7}\sqrt{10}$.

(b) (ii) & (iii) $\Rightarrow AE : EC = \frac{\frac{6}{7}\sqrt{10}}{\frac{10}{7 \sin\theta}} = 9 : 5$

$\frac{6 \sin\theta}{\sqrt{10}} = \frac{9}{5}$

$\Rightarrow \sin\theta = \frac{3\sqrt{10}}{10}$

$\Rightarrow sin(tan^{-1} 3) = \frac{3}{\sqrt{10}}$ ….. by (b) (i).

After finishing the above worksheet, the teacher tells the students that the original style of the HKCEE problem is actually like this:

(Original style of a modified HKCEE multiple choice question)

Refer to the diagram shown below (see Figure 2). *ADB*, *AEC*, *BCF* and *DEF* are straight lines. Given that $AD = DB$ and $DE : EF = 2 : 5$, find the ratio of $AE : EC$.

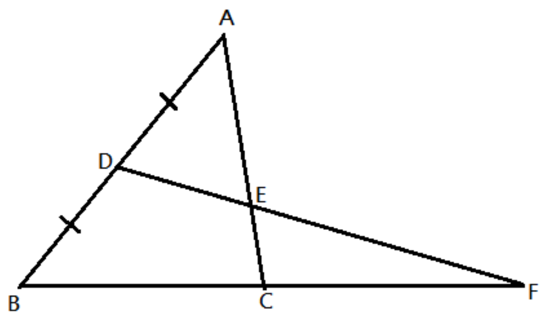


Figure 2

Teacher informs the students that such a problem can be solved from multiple perspectives even if we remove the coordinates framework embedded in the previous worksheet. Then students will be given time for trying it out and discussion could be made afterwards. Finally, the following suggested solutions could be displayed for students’ appreciation.

Method 1

Draw DG such that DG // BF as shown below (see Figure 3).

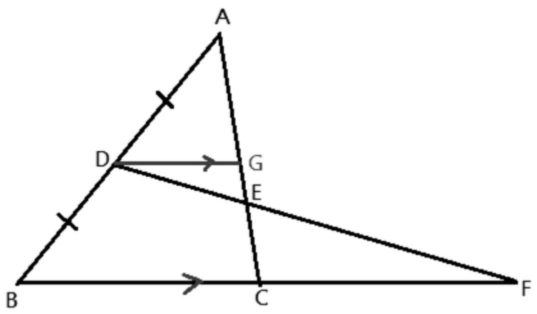


Figure 3

$AD = DB$ (given),

DG // BC (construction),

$AG = GC$ (intercept theorem).

Note that $\Delta DGE \sim \Delta FCE$ (AAA) and $DE : EF = 2 : 5$ (given), we have

$GE : EC = DE : EF = 2 : 5$ (corr. sides, ~Δs).

Hence, $AE : EC = [(2+5)+2] : 5 = 9 : 5$.

Method 2

Draw EG such that EG // AB as shown below. (see Figure 4)

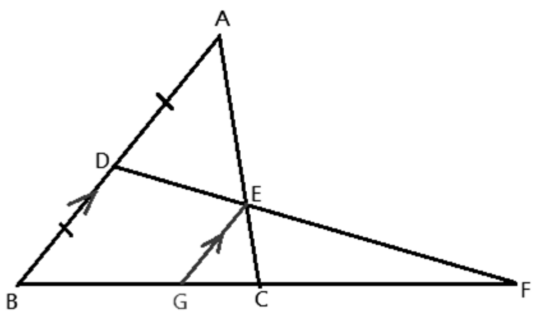


Figure 4

$FE : ED = 5 : 2$ (given) $\Longrightarrow$ $FE : FD = 5 : 7$,

Note that $\Delta FEG \sim \Delta FDB$ (AAA), we have $EG : DB = FE : FD = 5 : 7$ (corr. sides, ~Δs).

Let $EG = 5k$ and $DB = 7k$ where $k \neq 0$.

$\because$ $AD = DB$ (given), $\therefore$ $AB = 14k$.

Also, $\Delta ECG \sim \Delta ACB$ (AAA),

we have $\frac{EG}{AB} = \frac{CE}{CA} = \frac{5k}{14k} = \frac{5}{14}$ (corr. sides, ~Δs).

Hence, $\frac{EC}{EC+AE} = \frac{5}{14} \Longrightarrow 14EC = 5EC + 5AE \Longrightarrow 9EC = 5AE \Longrightarrow \frac{AE}{EC} = \frac{9}{5}$.

i.e. $AE : EC = 9 : 5$.

Method 3

Draw *CD* and *AF* as shown below (see Figure 5).

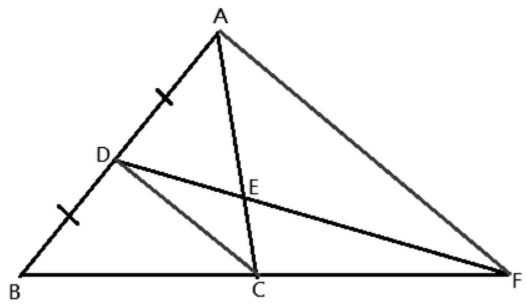


Figure 5

$\because DE : EF = 2 : 5$ (given),

$\therefore$ $\frac{area\ of\ \Delta ADE}{area\ of\ \Delta AEF} = \frac{DE}{EF} = \frac{2}{5}$. (trick of triangles having same height)

Similarly, $\frac{area\ of\ \Delta CDE}{area\ of\ \Delta CEF} = \frac{2}{5}$.

Let area of $\Delta ADE = 2a$ and area of $\Delta CDE = 2b$ respectively where $a \neq 0$ and $b \neq 0$.

Then, area of $\Delta AEF = 5a$ and area of $\Delta CEF = 5b$.

$\because$ AD = BD (given),

$\therefore$ area of ΔBDC = area of $\Delta ADC = 2a + 2b$. (triangles having same bases and same heights)

Similarly, we have area of ΔADF = area of ΔBDF

$\Rightarrow 2a + 5a = (2a + 2b) + 2b + 5b$

$\Rightarrow 5a = 9b$.

So, $AE : EC = 5a : 5b = 9b : 5b = 9 : 5$.

Method 4

Draw BE and AF as shown below (see Figure 6).

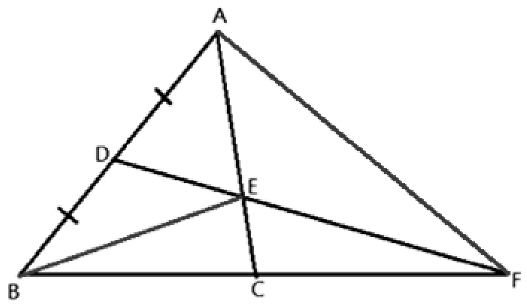


Figure 6

$\because DE : EF = 2 : 5$ (given),

$\therefore \frac{area\ of\ \Delta ADE}{area\ of\ \Delta AEF} = \frac{DE}{EF} = \frac{2}{5}$ (trick of triangles having same height).

$\because$ AD = BD (given),

$\therefore$ area of ΔADE = area of ΔBDE (triangles having same bases and same heights).

Hence, area of ΔABE : area of ΔADE = $4 : 5$.

As a result, $BC : CF = 4 : 5$.

Let the area of $\Delta AEF = 5k$ where $k \neq 0$.

Then, area of ΔBEF = area of $\Delta AEF = 5k$ (triangles having same bases and same heights).

So, area of $\Delta ECF = 5k \times \frac{5}{4+5} = \frac{25k}{9}$.

Thus, $AE : EC$ = area of ΔAEF : area of ΔECF = $5k : \frac{25k}{9}$ = $9 : 5$.

Reflection

The purpose of the reframed HKCEE problem can be two-fold. On one hand, it gives a good revision opportunity for junior form students on the topic of coordinate geometry. On the other hand, it presents an indirect way of proving $sin(tan^{-1}3) = \frac{10}{\sqrt{3}}$ as well as a possible strategy of solving the original HKCEE multiple choice question (i.e. finding the ratio of $AE : EC$).

Note that Method 1, Method 2, Method 3 and Method 4 depicted above represent some other strategies to be supplemented by the teacher for solving the original HKCEE multiple choice question. Comparing Methods 1 and 2, they both employ the junior form techniques of similar triangles and adding suitable parallel lines. But the way how we add the parallel line may not be obvious to some students. Method 3 and Method 4 have the key strength that only one single technique about area of triangle is required and such technique is elementary to many students though two lines have to be added in the figure. However, the idea of Method 4 should be more difficult to think of compared with Method 3. With all these possible solutions in hands, teacher has more rooms to discuss with students and it is expected that the higher order thinking skills of students could be boosted consequently.

The readers may wonder if we could reframe the selected HKCEE problem in another way. The answer is certainly yes and we have worked out another "reframed version" tailor-made for students taking extended module 2. The details are summarized as follows:

(Version 2 of reframing a modified HKCEE multiple choice question)

Refer to the diagram shown below (see Figure 7). *ADB*, *AEC*, *BCF* and *DEF* are straight lines. Given that $AD = DB = 1$, $DE = 2$ and $EF = 5$. Denote $AE = x$ and $EC = y$, find the ratio of $x : y$.

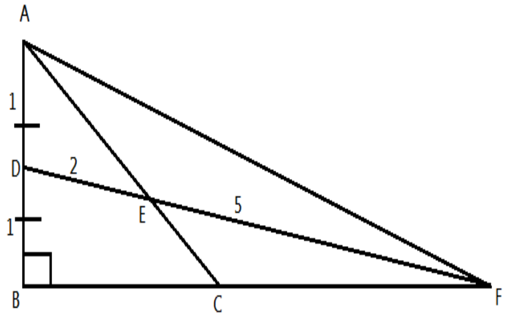


Figure 7

(a) (i) Find the lengths of *BF* and *AF*.

Solution

$BF^2 + 1^2 = 7^2$ (Pyth. Theorem),

$BF = \sqrt{48}$.

$AF^2 = 2^2 + \left(\sqrt{48}\right)^2$ (Pyth. Theorem), .

$AF = \sqrt{52}$.

(ii) Let $\angle AFD = \alpha$ and $\angle DFB = \beta$. Find the values of $\cos\alpha$ and $\sin\beta$.

Solution

By cosine formula, $1^2 = 7^2 + \left(\sqrt{52}\right)^2 - 2(7)\left(\sqrt{52}\right)\cos\alpha$.

$\therefore \cos\alpha = \frac{25}{7\sqrt{13}}$.

$\sin\beta = \frac{DB}{DF} = \frac{1}{7}$.

(b) Find the value of x .

Solution

By Cosine formula, $x^2 = \left(\sqrt{52}\right)^2 + 5^2 - 2(5)\left(\sqrt{52}\right)\cos\alpha$.

$\Longrightarrow x^2 = 77 - 10\sqrt{52} \times \frac{25}{7\sqrt{13}} = 77 - \frac{500}{7} = \frac{39}{7}$.

i.e. $x = \frac{\sqrt{273}}{7}$.

(c) Let $\angle AEF = \gamma$ and $\angle ACF = \theta$.

(i) Find $\cos\gamma$ and $sin\,\theta$.

Solution

By Cosine formula, $\left(\sqrt{52}\right)^2 = 5^2 + \left(\frac{\sqrt{273}}{7}\right)^2 - 2(5)\left(\frac{\sqrt{273}}{7}\right) cos\,\gamma$.

Now, $\gamma = \theta + \beta$ (ext. ∠ of Δ) and thus $\theta = \gamma - \beta$.

By compound angle formula, we have

$sin\,\theta = sin(\gamma - \beta) = sin\,\gamma\, cos\,\beta - cos\,\gamma\, sin\,\beta$

$= \sqrt{1 - \left(-\frac{15}{\sqrt{273}}\right)^2} \times \frac{\sqrt{48}}{7} - \left(-\frac{15}{\sqrt{273}}\right) \times \frac{1}{7} = \frac{9}{\sqrt{273}}$.

(ii) Hence, find the value of y .

Solution

By sine formula, we have $\frac{5}{\frac{9}{\sqrt{273}}} = \frac{y}{\frac{1}{7}} \Rightarrow y = \frac{5\sqrt{273}}{63}$.

(d) Find $x : y$.

Solution

$x : y = \frac{\frac{\sqrt{273}}{7}}{\frac{5\sqrt{273}}{63}} = \frac{9}{5} = 9 : 5$.

Finally, the teacher may ask students to check the relationship between the above new reframed approach and the original modified HKCEE multiple choice question. Then, they will learn another additional trick so-called "take special example". Here, we force $\angle ABF = 90^\circ$ which is just a special example for the diagram shown in the original modified HKCEE multiple choice question.

Concluding Remarks

In this paper, we present totally six different approaches (direct or indirect) to solve a modified HKCEE multiple choice question. More importantly, we introduce the idea of "reframing" so as to make our teaching flows to be more non-linear. As a final remark, teacher may also try asking students to tell their own mathematical story for proving $sin(tan^{-1} 3) = \frac{10}{\sqrt{3}}$. This will certainly improve the creativity of our students!

Acknowledgement

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References

HKCEE 1996 (Paper 2, Q52), Hong Kong Examinations and Assessment Authority.

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