A Reflection on a HKDSE Mathematics Problem

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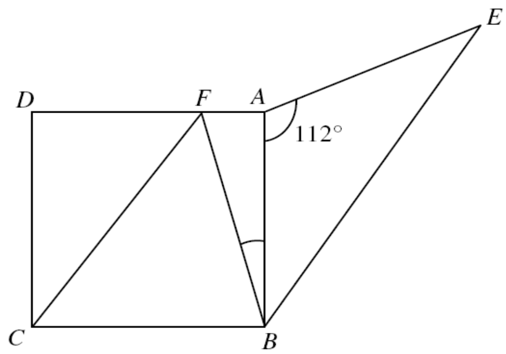
Introduction

Recently, my S6 students have completed their HKDSE mock examination at school. The paper that they were attempting was actually the practice paper (Mathematics, Compulsory Part) offered by HKEAA. I found that there was an interesting question from the MC paper 2 .

In the figure, *ABCD* is a square. *F* is a point lying on *AB* and *BE*//*CF*. If *AB* = *AE*, find the size of $\angle ABF$. Correct your answer to the nearest degree.

A. 17°
B. 18°
C. 23°
D. 28°



HKDSE Practice Paper Compulsory Part Paper 2 Question (Modified)

I anticipated that some students may have trouble in solving this problem and thus I tried to think of various strategies to tackle this problem. My original purpose was simply to let my students appreciate the beauty of mathematics. Just like what the subject "Liberal Studies" always emphasize, we should encourage the students to think of the problem from multiple perspectives. Hereafter, I will present five possible methods for solving the problem.

Method 1 (Accurate Reconstruction of the Figure)

For this problem, a quick method is to re-draw the Figure 1 using ruler and set squares (better with large size) on a piece of rough worksheet. In doing so I found that the answer is closer to 18^{o} rather than 17^{o}. Hence, the correct answer should be option B.

Method 2 (Standard)

The following method is a popular one and one of my students used this method.

Let the length of each side of the square be x and the required angle be θ.

In ΔABE, $\angle ABE = \angle AEB$ (base $\angle$s, isos. Δ). (Refer to Figure 1)

$2\angle ABE + 112^{o} = 180^{o}$ ($\angle$ sum of Δ)

So we get $\angle ABE = 34^{o}$.

$\angle DCB = \angle ABC = 90^{o}$ (Property of Square).

$\angle FCB + \angle CBE = 180^{o}$ (int. $\angle$s, CF // BE)

$\therefore$ $\angle FCB = 180° - (90° + 34°) = 56°$.

Thus, $\angle DCF = 90° - 56° = 34°$.

In ΔDCF, $DF = x\tan 34°$.

In ΔABF, $\tan\theta = \dfrac{x - x\tan 34°}{x} = 1 - \tan 34°$

i.e. $\theta = \tan^{-1}(1 - \tan 34°) \approx 18.0°$

Method 3 (Area)

The basic idea of this method is to consider the areas.

Area of ΔDCF = $\dfrac{1}{2}x(x\tan 34°) = \dfrac{1}{2}x^2\tan 34°$.

Area of ΔABF = $\dfrac{1}{2}x^2\tan\theta$.

Since area of square = area of ΔDCF + area of ΔCFB + area of ΔABF,

we have $x^2 = \dfrac{1}{2}x^2\tan 34° + \dfrac{1}{2}(x)(x) + \dfrac{1}{2}x^2\tan\theta = \dfrac{x^2}{2}\tan 34° + \dfrac{x^2}{2} + \dfrac{x^2}{2}\tan\theta$ (*).

By cancel out x^2 (since $x \neq 0$) and rearranging the terms in (*),

We have $\tan\theta = 1 - \tan 34°$ and the result follows.

Method 4 (Trial and Error)

What if students got no clue in doing an MC question? A very popular trick is to adopt what is called the "reverse thinking methodology". In other words, we can try out the options one by one and see which one fits the most. We firstly assume that option A is correct. Then, we take $\angle ABF = 17°$. As a result, $\angle CFB = \angle ABF + \angle ABE = 17° + 34° = 51°$ (alt. $\angle$s, $CF // BE$)

Now, we focus on ΔBCF. If the option A is really correct, the sine law should hold. However, don't forget that the options provided are not exact values. We can only compare the error instead.

In ΔCDF, $CF = \dfrac{x}{\cos 34°}$

In ΔBCF, we have $\dfrac{BC}{\sin \angle CFB} = \dfrac{CF}{\sin \angle CBF} = \dfrac{BF}{\sin \angle BCF}$ (by sine law).

we have $\dfrac{x}{\sin 51°} \approx \dfrac{\frac{x}{\cos 34°}}{\sin(90° - 17°)} \Rightarrow \dfrac{1}{\sin 51°} \approx \dfrac{1}{\cos 34° \sin 73°}$.

Define error_A = $\dfrac{1}{\sin 51°} - \dfrac{1}{\cos 34° \sin 73°}$ Then error_A ≈ 0.02543.

Similarly, if we assume option B is the correct answer,
then we take $\angle ABF = 18°$ and so $\angle CFB$ = $18° + 34° = 52°$.

Therefore, we have $\dfrac{x}{\sin 52°} \approx \dfrac{\frac{x}{\cos 34°}}{\sin(90° - 18°)} \Rightarrow \dfrac{1}{\sin 52°} \approx \dfrac{1}{\cos 34° \sin 72°}$.

The new error term, error_B = $\dfrac{1}{\sin 52°} - \dfrac{1}{\cos 34° \sin 73°} \approx 0.00073 <$ error_A.

Careful readers should check error_C and error_D (errors for options C and D respectively) to confirm that option B is the best answer.

Method 5 (Analytic Approach)

Another cross-over solution is to assign the coordinates system to the figure and then solve it accordingly. See the diagram below for the settings:

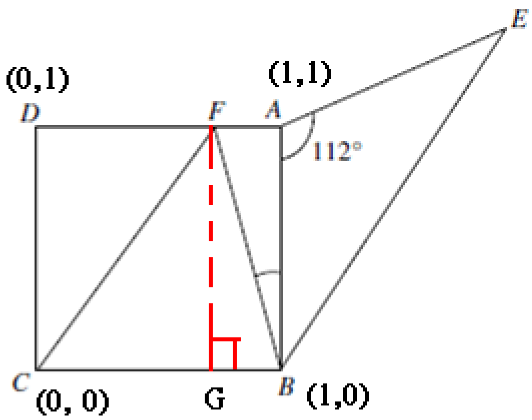


Figure 2

More precisely, we consider point C as the origin. As all squares are similar, we may scale down the whole figure (maintain $\angle EAB$ = $112°$) so that the square becomes a unit square. Then, B = (1,0), A = (1,1) and D = (0,1). (see Figure 2)

As what we have proved before, $\angle FCB = 56°$ and $CF = \dfrac{1}{\cos 34°}$ which in turns tell us that the coordinates of $G = \left(\dfrac{\cos 56°}{\cos 34°}, 0\right)$ and $F = \left(\dfrac{\cos 56°}{\cos 34°}, 1\right)$. Therefore, $AF = 1 - \dfrac{\cos 56°}{\cos 34°}$.

Finally, by considering ΔABF, we have $\tan\theta = \dfrac{1 - \dfrac{\cos 56°}{\cos 34°}}{1}$ so that θ can be found.

Reflection

To this end, I would like to reflect on some features of the different solutions and some insights on teaching.

Method 1 is straightforward and fast, with a major drawback that it may lead to huge errors due to drawings of the figure or the measurement.

Method 2 is a standard method that a S2 student may be able to use.

Method 3 is innovative by considering the relationships between areas. However, it is more or less the same as method 2. In fact, it gives no better efficiency in solving the problem.

Method 4 can be regarded as performing error analysis. It is a kind of 'reverse thinking methodology' in which a very rigid solution is not required.

Method 5 is in fact a popular way of designing a question for public examination. We may refine the problem as a long question. Say, part (a) asks the students to write down the coordinates of F in terms of sine or cosine ratios whereas part (b) guides the students to find θ by considering ΔABF. In particular, this method also refreshes students' memories in dealing with deductive geometry by introducing a coordinate system to the diagram.

I have discussed with Dr. Allen Leung (my M.Ed. lecturer from the Hong Kong Baptist University) to explore further this problem. He pointed out that $\tan\gamma + \tan\theta = 1$ for any point F lying on any position of the side AD of a square, where $\angle DCF = \gamma$ and $\angle ABF = \theta$. (See Figure 3 shown below)

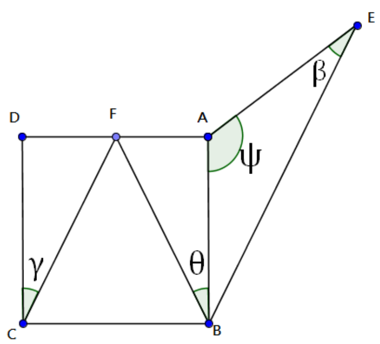


Figure 3

The idea can simply be explained in this way. For the ease of presentation, we may assume that $ABCD$ is a unit square.

Then, we have $DF = \tan\gamma$ and $AF = \tan\theta$.

$DF + AF = AD \Rightarrow \tan\gamma + \tan\theta = 1$. ----- (*)

Another related discovery is that, in the original DSE practice paper problem, there is another angle ψ (i.e. $\angle BAE$). Note that this angle ψ is not formed freely but it has to be formed such that *CF* // *BE* and *AB* = *AE*. Grounded with these constraints, the relation (*) can be adjusted and restated as follows:

$$\tan\beta + \tan\theta = 1 \quad (\because \beta = \gamma)$$

$$\tan\left(\frac{180^\circ - \psi}{2}\right) + \tan\theta = 1 \quad (\because \angle DCF = \angle ABE = \angle AEB\,; \quad \text{and } \psi + 2\beta = 180^\circ)$$

$$\tan\theta = 1 - \tan\left(90^\circ - \frac{\psi}{2}\right) = 1 - \frac{1}{\tan\left(\frac{\psi}{2}\right)} \quad \text{---- (\#)},$$

which turns out to be another interesting relationship. Imagine that if students know this formula (#) in advance, the required answer of θ can be found at once!

Concluding Remarks

In summary, a key to solve this DSE problem is to realize that $\angle DCF = \angle ABE$ and there are many possible methods of solving the problem. It is worth noting that, however, the underlying relationship (*) and (#) as depicted in the reflection part are both beautiful enough for students to appreciate the beauty of mathematics. If so, why don't we let students see more in our teaching? The remaining question is, how could we introduce these concepts to the students in a more natural and constructive manner? With a view to letting students know the relationship of $\tan\gamma + \tan\theta = 1$, the following describes a possible teaching approach / activity for the DSE problem:

Use GeoGebra (a dynamic software) to construct the required diagram shown in the DSE problem. (See Figure 4)

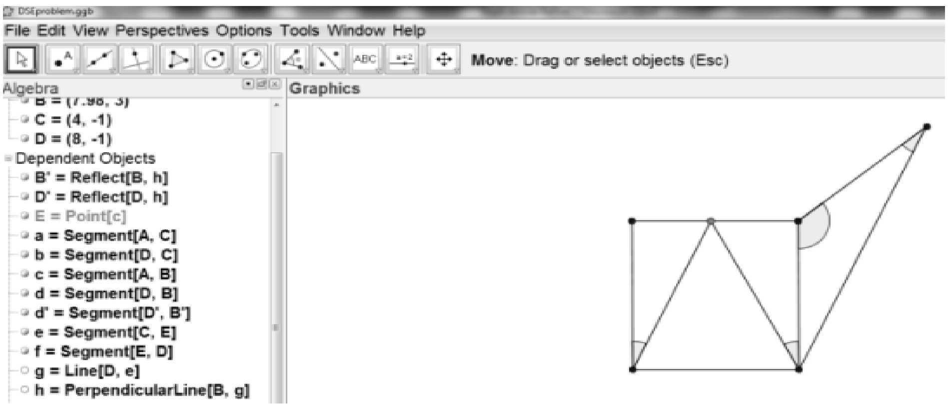


Figure 4

Vary a key feature of the diagram but keeping all the other features to be fixed. In this way, students could easily see the interaction between different components in the diagram and focus on the changes in the scenario. Based on this rationale, we may drag the point *F* along the side of the square and discover the persistent result

$\tan\gamma + \tan\theta = 1$. (See Figure 5a and 5b for instance)

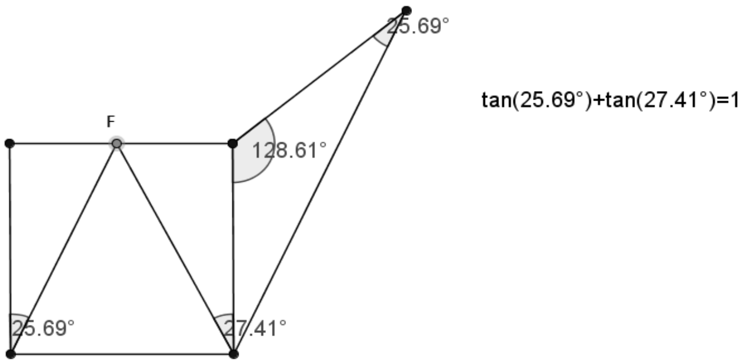


Figure 5a

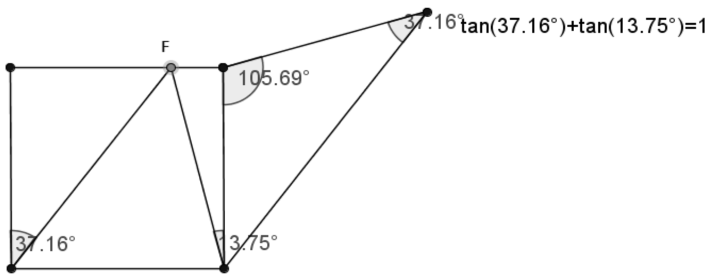


Figure 5b

To stimulate students' logical thinking, we could also ask students what will happen if the point *F* is moved to the rightmost corner. (See Figure 6)

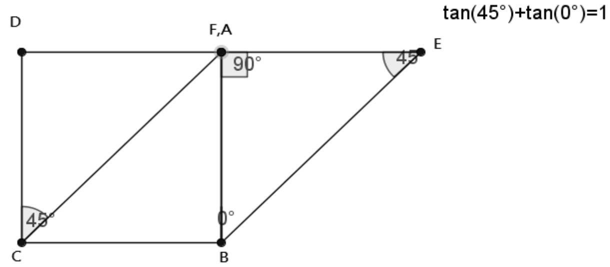


Figure 6

When F moves to the right, $\angle FBA = \theta$ decreases and eventually becomes $0°$ when the points F and A coincide. At this moment, students may quickly verify the result that $\angle DCF = \angle EBA$ as they both become $45°$ and the triangles FDC and EAB become two identical right-angled isosceles triangles.

Acknowledgement

Special thanks go to Dr. Allen Leung for his stimulating lectures on the Theory of Variation while I was attending his M.Ed. course “Theory, Research and Pedagogical Issues of Shape and Space” at Hong Kong Baptist University. Also, I would like to thank my students who were willing to explore deeper when I was discussing with them for this problem.

References

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