The Mathematics of Woodturning

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Introduction

Woodturning is a very popular hobby in the UK, with many woodturning societies and magazines, an incredible variety of tools supporting many variations of woodturning. You cannot visit a craft fair without finding at least one stall selling a collection of items lovingly turned from wood. A skilled craftsman will take pride in telling you the ins and outs of the types of wood and the best way to perfect the polish. However, the one constant among all these variations are the mathematical characteristics common to all woodturned objects.

Woodturning provides a very practical and visual introduction to Volumes of Revolution and from there to Cylindrical Polar Coordinates. We will be using Autograph (www.autograph-maths.com) to visualize the mathematical concepts in this article.

Woodturning

All you need to begin woodturning is a lathe and a variety of chisels. You start with a piece of wood mounted between the headstock and the tailstock. The piece of wood is called a wooden spindle. The headstock is connected to a motor which controls rotation. A toolrest is positioned parallel to the wooden spindle. A chisel is placed on the toolrest and applied to the turning wood. At first to remove stock quickly, then to shape it, plane it smooth, and decorate it.

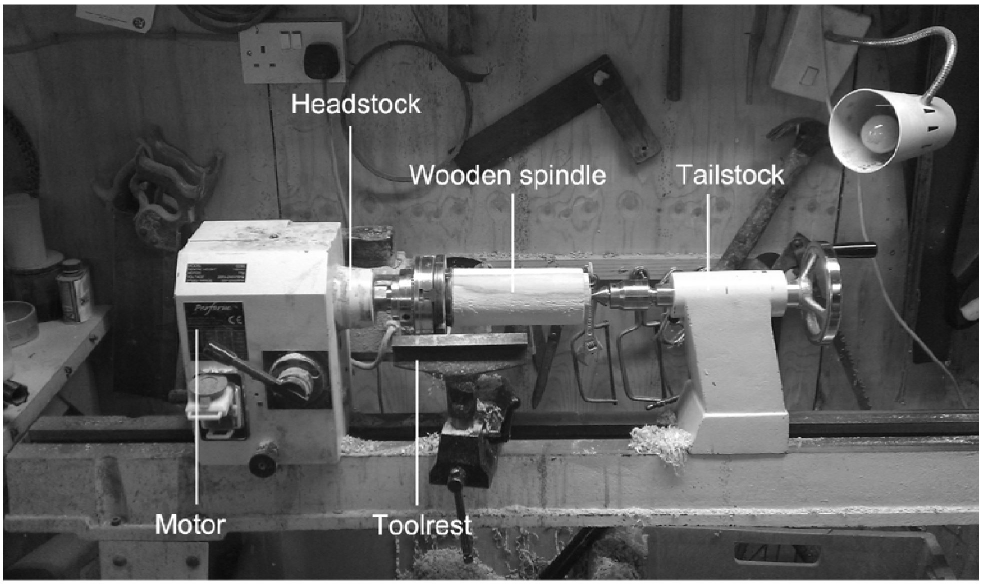

Figure 1: Parts of a Lathe

Examples of woodturned items include table legs, pens, candlesticks, chess pieces, woodwind musical instruments, lamps, toys and decorative items.

Figure 2: Woodturned items

Volume of Revolution

The straight line connecting the centres of the headstock and the tailstock is the axis about which the wooden spindle rotates. Now imagine a plane that passes through this axis and the line created by the toolrest. The chisel is applied to the wooden spindle by moving it towards the axis in this plane. As the chisel cuts into the wooden spindle it traces out a curve as it moves left to right.

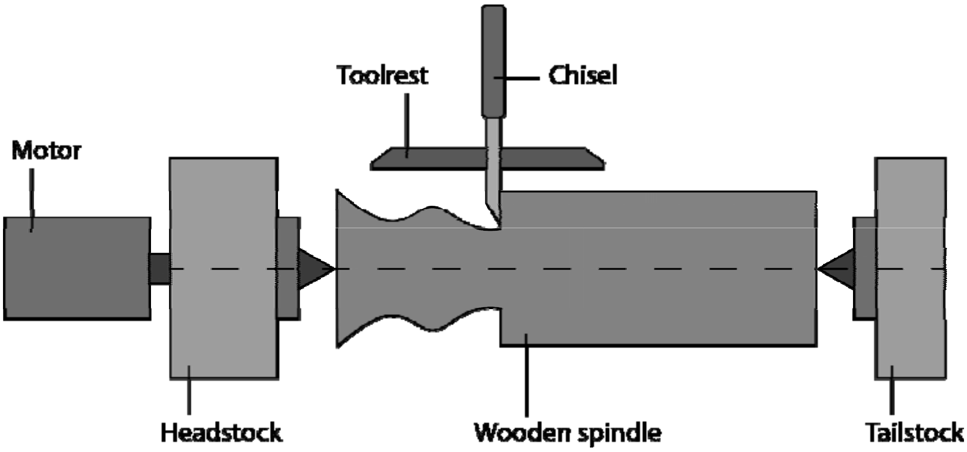

Figure 3: Lathe diagram

The wooden spindle rotates about its centre as the chisel is applied, so the chisel is the same distance from the centre of rotation regardless of the angle of rotation. If we were to make a cut at any point perpendicular to the axis of rotation we would find that the cross-section is a circle.

Consider the area between the curve and the axis of rotation. The shape of the wooden spindle is the volume through which this area passes as it is rotated about the axis. This is known as a volume of revolution.

Modelling Woodturned Objects with Autograph

Figure 4: Woodturned vase made by Bob Guy

If you do not already have the latest version of Autograph 3.3.10, download a free trial from: www.autograph-maths.com/download

Open a New 2D Graph Page.

Select Equal Aspect Mode.

Click and drag the following image onto the Autograph page.

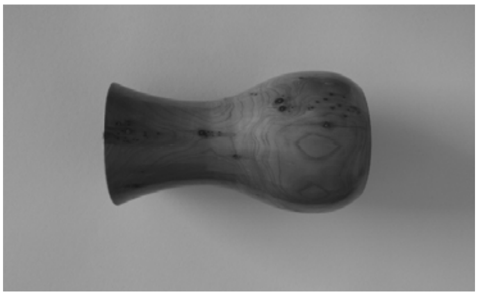

Figure 5: Woodturned vase horizontal (click and drag into Autograph)

(Alternatively save the image to your computer and then, in Autograph, right-click, choose Insert Image, and browse to the image on your computer.)

Double-click on the image and increase the transparency to 50% so you can see the axes through the image. Then drag the image so the centre of rotation lines up with the *x*-axis.

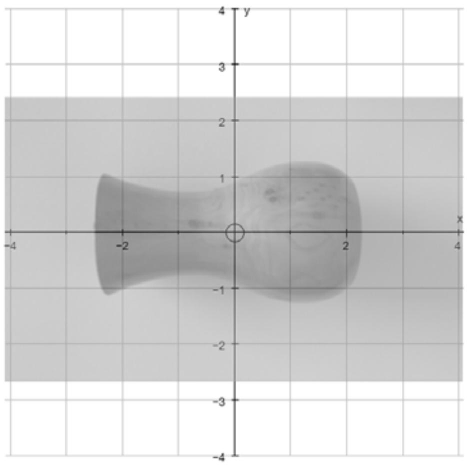

Figure 6: Woodturned vase image in Autograph

The task now is to find an equation which describes the curve of the top of the object. I decided to try a sine wave but you can choose any equation you like at this stage.

Enter the equation: $y = a\sin(bx + c) + d$.
Click Edit Constants and change the values of c and d to 0, leaving the values of a and b as 1. So initially this equation is $y = \sin(x)$.

Select the curve and click Text Box, this will add a box which shows the equation and the values of the constants a , b , c and d .

The equation $y = \sin(x)$ is not a good approximation to the top of the object, we need to make some adjustments to the constants a , b , c and d to see if we can make it a better fit. This is a great opportunity to tests students' understanding of the effect of the different constants in this equation. Remember to ask them to predict what will happen before making any changes.

Open the Constant Controller. From the drop-down menu, choose the constant you would like to change. Use the up and down arrow to change the value of the constant, and the left and right arrows to change the amount by which the constant is changed. Continue until you have a good fit.

Add two points to the curve at either end of the object.

Select both points, right-click and choose Find Area. Make a note of the Start Point and End Point parameters, which are determined by the position of the points we added in the last step, we will be using these values again shortly. Set the number of divisions to 500 and click OK. The area that we will be rotating about the *x*-axis is highlighted.

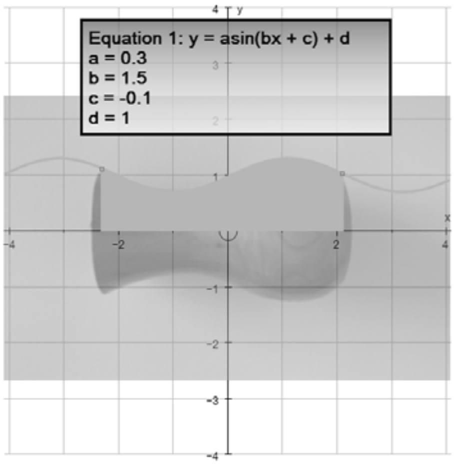

Figure 7: Woodturned vase curve modelled in Autograph

Note: In a classroom situation you may like to take more time to explain area approximations. By starting with a small number of divisions and using the Animation Controller and Zoom Tools to gradually increase the number of divisions you can illustrate the concept of calculus. The focus of this article is to describe a practical application, therefore we will not be describing all of these steps here. See www.autograph-maths.com/videotutorials for more information.

Open a New 3D Graph Page.

Select *x*-*y* Orientation. (You will need to click the little black arrow to the right of .)

Enter the equation: $y = a\sin(bx + c) + d$.
Tick Plot as 2D Equation.
Click Edit Constants and change the values of a, b, c and d to match the values you found earlier. For me $a = 0.3$, $b = 1.5$, $c = -0.1$ and $d = 1$.

Select the curve, right-click and choose Find Area. Select Simpson's Rule because it is a better approximation and results in a smoother volume of revolution. Set the Start Point and End Point as above (for me Start Point = –2.3 and End Point = 2.1), and set the number of Divisions to 50.

Turn on Slow Plot Mode.

Select the area, right-click and choose Find Volume.

Click and drag to move the object around.

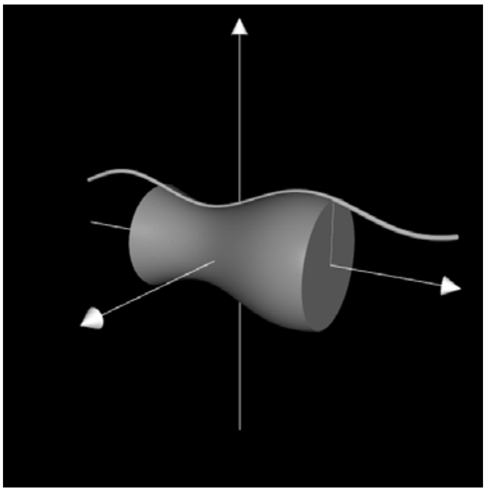

Figure 8: Woodturned vase volume of revolution in Autograph

The volume of revolution is a good approximation to the woodturned object.

The following figure shows some other examples of woodturned objects which I have analysed in the same way.

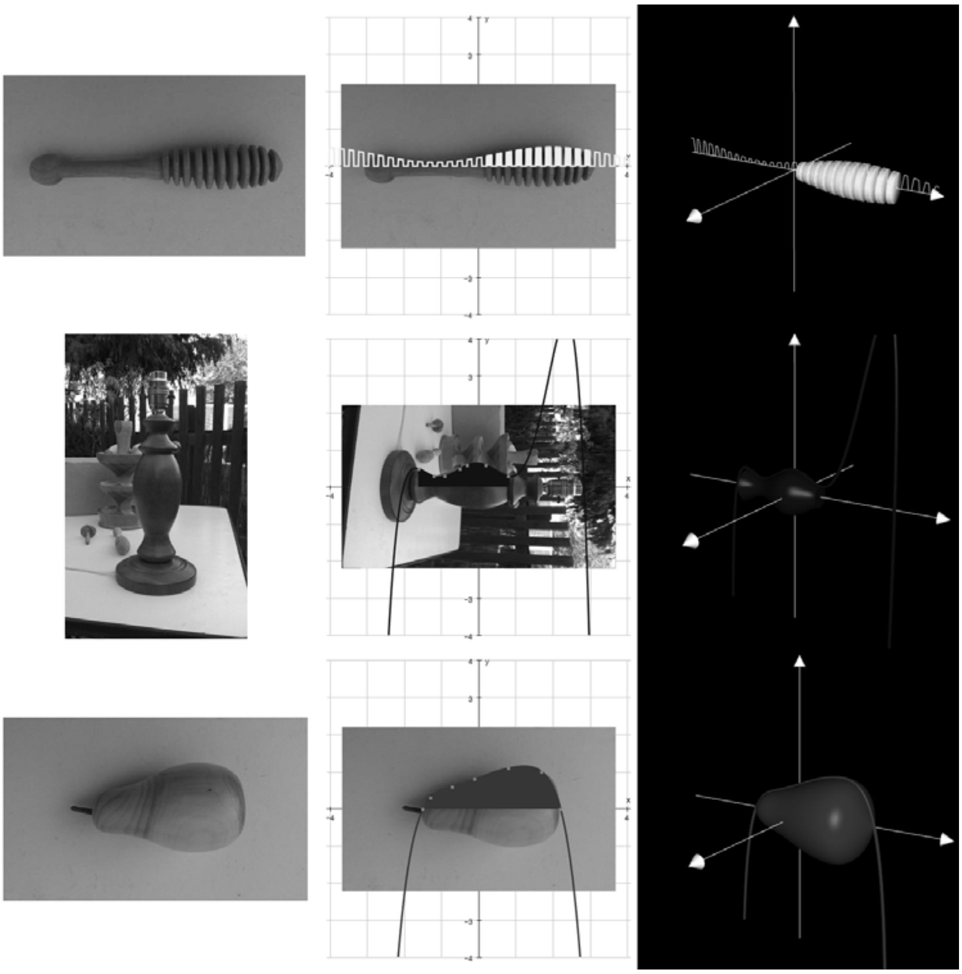

Figure 9: Woodturned honey dipper, lamp and pear all made by Bob Guy and modelled in Autograph

Search the internet for images of other woodturned objects and try modelling them in Autograph. Go to Help > Autograph Help > Search and enter "Best Fit", there you will find a description of other ways of fitting curves.

Cylindrical Polar Coordinates

Woodturning is clearly a very good example of volumes of revolution and we have shown how to recreate a turned wooden spindle mathematically by modelling the edge with a curve $y = f(x)$, finding the area under this curve, and rotating it about the x-axis. However, whilst this gives us a method for recreating the shape it does not provide us with the mathematical equation of the surface of the whole shape. Fortunately this is very easy if we use cylindrical polar coordinates.

In three-dimensions the position of a point P can be described in Cartesian coordinates, by the distance from the origin along the three perpendicular axes x, y and z. Alternatively the position of the point P can be described in cylindrical polar coordinates, by the distance along the z-axis z, the distance from the z-axis r, and the rotation about the z-axis θ.

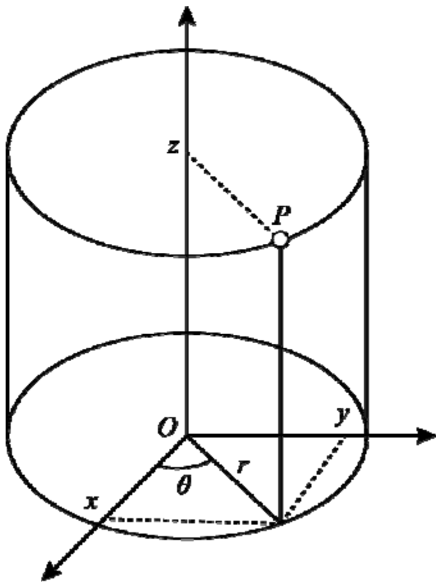

Figure 10: Cylindrical polar coordinates

Which coordinate system you use is up to you, sometimes one is more appropriate than another. For example the equation of a cylinder of radius 1 in is $x^2 + y^2 = 1$ in Cartesian coordinates, whereas in cylindrical polar coordinates the equation is $r = 1$, which is much neater.

Cylindrical polar coordinates are particularly good for describing the equation of the surface formed by rotating a curve about the z-axis. Let the axis of rotation of the wooden spindle be the z-axis. Then the distance along the axis of rotation is z, the distance of the chisel from the axis of rotation is r, and the angle of rotation is θ.

The distance of the chisel from the axis of rotation, r, changes with the distance along the axis of rotation, z, but not with the angle of rotation, θ. (Remember that if we were to make a cut at any point perpendicular to the axis of rotation we would find that the cross-section is a circle.)

In our earlier analysis: the axis of rotation was the x-axis, y was the distance from the axis of rotation, x was the distance along the axis of rotation, and we found a function which describes the curve of the top of the object $y = f(x)$ (for the vase example $y = 0.3\sin(1.5x - 0.1) + 1)$.

Now: the axis of rotation is the z-axis, r is the distance from the axis of rotation, z is the distance along the axis of rotation, and so by comparison $r = f(z)$ (for the vase example $r = 0.3\sin(1.5z - 0.1) + 1)$. However, there is one significant difference between these two parameterizations. In the first case, y is the distance from the axis of rotation *in one particular direction*, which is why we must rotate about the axis to form the shape. In the second case, r is the distance from the axis *in any direction*, so the equation $r = f(z)$ describes the whole surface.

Plotting the Surface of Woodturned Objects with Autograph

Open a New 3D Graph Page.

Enter the equation: $r = a\sin(bz + c) + d$.

Click Edit Constants and change the values of a , b , c and d to match the values you found earlier. For me $a = 0.3$, $b = 1.5$, $c = -0.1$ and $d = 1$.

Click Startup Options, untick Use z-axis ranges, and set z-start and z-finish to the Start Point and End Point values you found earlier. For me z-start = –2.3 and z-finish = 2.1 .

Click and drag to move the object around.

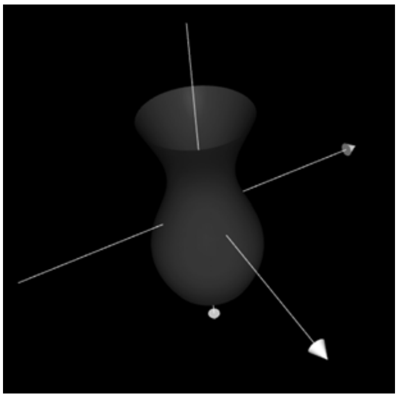

Figure 11: Woodturned vase in cylindrical polar coordinates

Extension: The Rose Engine Lathe

Of the many variations on the art of woodturning, the Rose Engine Lathe provides a very nice mathematical extension to our activity.

Figure 12: Rose engine lathe (image courtesy of John Edwards)

Rose Engine Lathes look more like expensive pieces of furniture than industrial machines, and this is because they were. Before the introduction of the motor car wood turning was the hobby of the gentry, back in 1850 one geometric lathe sold for £1500 – at a time when the average price of a house was only £25, this was a considerable sum of money.

The mechanical difference which makes Rose Engine Lathes mathematically interesting is that the headstock and tailstock rock back and forth while the wooden spindle is rotated, and the toolrest remains stationary. This rocking motion means the distance of the chisel from the axis of rotation, changes as the angle of rotation changes. The cross-section of the wooden spindle is no longer required to be circular.

In the following example, the base of the woodturned object is made with a standard lathe, and therefore has a circular cross-section. The top of the piece is made with a Rose Engine Lathe and has more interesting cross-section.

Figure 13: Rose engine bowl by Paul Coker

Investigating Rose Engine Lathes with Autograph

For this part of the investigation we will focus on the top part of the Rose Engine Bowl, with the unusual cross-section. In this case we cannot describe the object using a volume of revolution because the cross-section is not a circle. However we can still provide an equation for the surface using cylindrical polar coordinates. We will begin by investigating the cross-section (how r depends on θ), then model the profile (how r depends on z), and finally bring these together to present an equation for the surface in cylindrical polar coordinates.

> Note: In a classroom situation we would expect a significant amount of investigation to take place in order to find suitable equations for modelling the cross-section and the surface of this object. This investigation has been omitted for brevity.

Open a New 2D Graph Page.

Select Equal Aspect Mode.

Turn on Slow Plot Mode.

Enter the equation: $r = 2$.

Autograph will slowly draw a circle of radius 2. This is the path of the chisel on a normal lathe, the distance of the chisel from the centre of rotation is constant.

Enter the equation: $r = 2 + |\sin(3\theta)|/2$.

Autograph will slowly draw the path that the chisel might take on a Rose Engine Lathe. The sine term produces a periodic hump. Can you find any other equations which give a similar pattern?

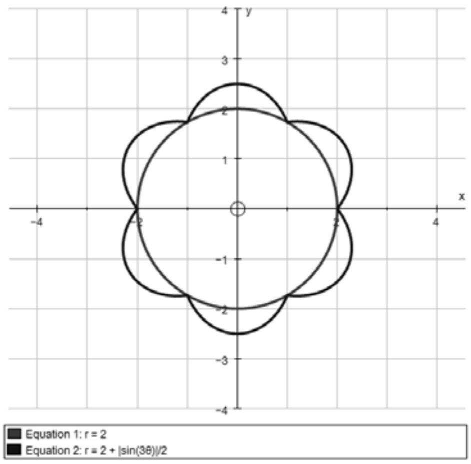

Figure 14: Rose engine bowl cross-section

Open a New 2D Graph Page.

Select Equal Aspect Mode.

Click and drag the following image onto the Autograph page.

Figure 15: Rose engine bowl horizontal (click and drag into Autograph)

(Alternatively save the image to your computer and then, in Autograph, right-click, choose Insert Image, and browse to the image on your computer.)

Double-click on the image and increase the transparency to 50% so you can see the axes through the image. Then drag the image so the centre of rotation lines up with the x-axis.

The task now is to find an equation which describes the curve of the top of the object. I decided to try a tanh wave but you can choose any equation you like at this stage.

Enter the equation: $y = a\tanh(bx)$.
The default values of a and b are 1. So initially this equation is $y = \tanh(x)$.

Select the curve and click Text Box, this will add a box which shows the equation and the values of the constants a and b .

Open the Constant Controller. From the drop-down menu, choose the constant you would like to change. Use the up and down arrow to change the value of the constant, and the left and right arrows to change the amount by which the constant is changed. Continue until you have a good fit.

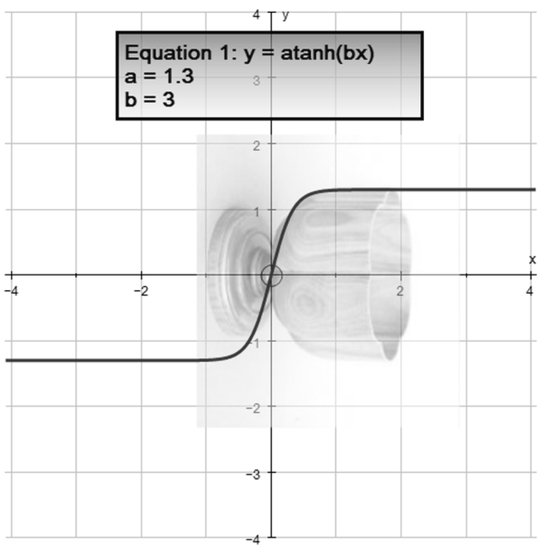

Figure 16: Rose engine bowl curve modelled in Autograph

Open a New 3D Graph Page.

Enter the equation: $r = 2 + |\sin(3\theta)|/2$.

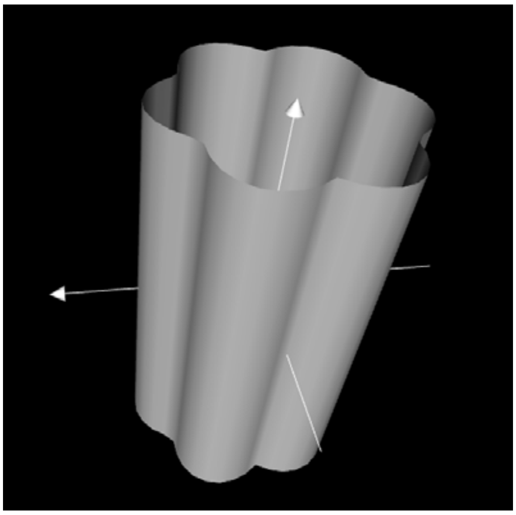

Figure 17: Rose engine bowl tube

This creates a tube with a rose cross-section. To add the profile of the rose engine bowl we need to add a component which represents the change in r due to z . We therefore multiply the equation by $\tanh(z)$, we could add some constants and adjust them to improve the approximation, but it is pretty good even without that added complication.

Edit the equation to be: $r = (2 + |\sin(3\theta)|/2)\tanh(z)$.
Click Startup Options, untick Use *z*-axis ranges, and set z-start = 0 .

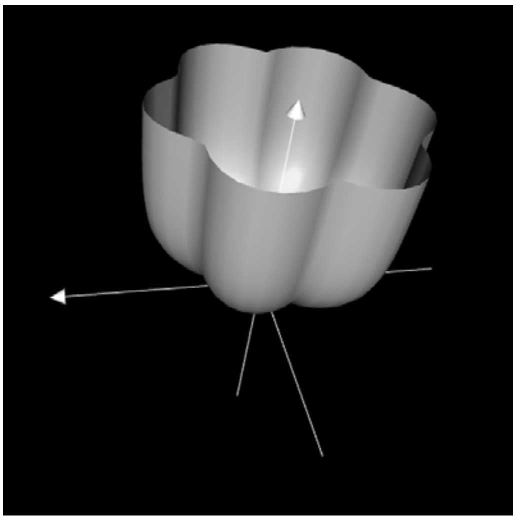

Figure 18: Rose engine bowl in cylindrical polar coordinates

Summary

In this article we have considered two methods of woodturning and used them to introduce the mathematical concepts of Volumes of Revolution and Cylindrical Polar Coordinates. Woodturning provides a tangible introduction to concepts which students sometimes struggle to visualise, and Autograph provides a simple tool to facilitate mathematical investigation of this subject. We have only scratched the surface in this article, there is a considerable amount of mathematical understanding to be gained by investigating woodturning in more depth. This might be the good basis for an independent student investigation.

Suggestions for further study include: hollow spindles, Ornamental Turning Lathes, Elliptical Cutting Frames, and turning on multiple axes. The follow sites are suitable for further reading:

John Edwards' Ornamental Turning: http://www.ornamentalturning.co.uk/
C. Paul Coker: http://www.cpaulcoker.co.uk/
The Society of Ornamental Turners: http://www.the-sot.com/
The Worshipful Company of Turners of London:
http://www.turnersco.com/

For more information on Autograph see the following sites:

Autograph Website: www.autograph-maths.com
TSM Resources: http://www.tsm-resources.com/autograph.html
Mr Barton Maths Videos:
http://www.mrbartonmaths.com/autographvideosmrb.htm

Acknowledgements

I would like to thank woodturners Bob Guy, Paul Coker and John Edwards for their valuable help in researching this article.