Centers of Data and Centers of Triangle

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This article suggests a comparison of two seemingly unrelated topics in our mathematics curriculum. The first topic is the study of various measures of centers of a data set in the Data Handling dimension. This topic is not particularly new in the current junior secondary curriculum (since 1999) but is associated with a few new or further emphasized learning objectives. The other topic, centers of triangle, is new in the Measure, Shape and Space dimension of the same curriculum. Both topics are quite often discussed among teachers as their objectives and emphases can be interpreted in different ways. Putting discussions of both topics together in this article may reveal some general questions about our pedagogical and curricular considerations.

Centers of Data

Mean, mode and median are common measures of center of a set of data, which could all be considered as average in different senses. To compare with the other topic in this article, I call them different centers, instead of different measures of center. In fact, they are representing the same set of data in different ways. Study of these averages is closely connected with other learning objectives of the same unit:

[a] construction of data set with given mean, mode or median

[b] characteristics of these centers

[c] how to choose and interpret a center

[a] is particularly interesting for there is no direct practical need for such "technique", unlike most of other items in this dimension. Some teachers or textbooks may treat it as a challenge for more able students after their mastery

of basic techniques, emphasizing problem solving and creativity. On the other extreme, there are strategies devised to help less able students to tackle such problems in a step-by-step manner. Are we really interested in certain techniques of constructing a data set or opening up a routine exercise in order to provide additional training in problem solving and creativity?

There is another way to understand this objective, though not necessarily the intention of the curriculum, relating to research focusing on concept of average and its development, such as those introduced in Shaughnessy (2006) and Watson (2006). Among these, the work of Russell and Mokros (1996) makes use of a simple diagnostic task that can be easily adopted for classroom teaching. The task requires students to think of some possible values that could be the prices of nine different brands of potato chips collected in a survey, while the average price is given. From the interviewees' responses, the researchers are able to identify some crucial conceptions of average among students regardless of their formal knowledge. In particular, they can distinguish the students' implicit assumptions, among others, of being "middle" or "most frequent" when talking about average (not necessarily because of any prior knowledge of mathematical definitions of median and mode).

An important implication from these studies is that mean, mode and median are not purely mathematical constructs and understanding of these centers can be built on learners' intuitive notions. Meanwhile, construction of data sets can be a useful task for students (regardless of ability or knowledge) to work on their intuitive understanding of average towards a mathematically precise distinction, facilitated by the teacher. Similar to the development of many mathematical ideas, it is possible to identify some intuitive and common conceptions of learners and make connections with rigorous mathematical formulations. In the case of average, common conceptions may lead to assumptions that a center should simultaneously (i) be the middle; (ii) represent the typical case (most frequently occurred); and (iii) be the result of even sharing, etc., as experienced in ordinary more or less unimodal symmetric data sets. The need to distinguish all these qualities as separate possible criteria of defining an average becomes sensible when learners can imagine all kinds of possible data sets in which the implicit assumptions are challenged. We can see then the objective [b] can be closely related to [a]. One common demonstration in [b] is using an extremely skewed data set to illustrate how far the mean and median can be separated. Considering both [a] and [b], there can be emphasis on developing learners' awareness of possible variation in data sets (in terms of shape of distribution) in order to appreciate precise mathematical formulations of average according to qualities we choose to focus on.

[b] is also further related to [c] in the sense that choice of a center depends on our (i) understanding of the meaning and characteristics of a center as well as (ii) understanding of the purpose of representing a data set in a particular context. Normally in our teaching, both aspects are considered and made explicit, to different extents, through examples created for discussing the choice of center. While being closely related, there could still be some distinction between [b] and [c]. As argued above, learning in [b] may focus on distinguishing the definitions, meanings and properties of these centers, usually with extreme cases to demonstrate the possibility of huge difference among these centers. In doing so, there should be no direct implication of choice of centers for representing the data without any consideration of context and purpose. For example, a mean, being above 99% of the data, does not indicate automatically its appropriateness as a summary of the data. What makes it problematic is our common conception that an average should be close to the middle value or a typical value among the data (having the characteristics of a median or a mode respectively).

Centers of Triangle

We now turn to the learning about centers of a triangle, but bear in mind the previous discussion about the concept of center and pedagogical implications. This topic is usually included in the later part of Key Stage 3, Measure, Shape and Space dimension, grouped together with other units in

Learning Geometry through Deductive Approach. Students are supposed to understand that there are special points associated with every triangle such as centroid, circumcenter and incenter, and they are related to special lines of a triangle. Some common questions among teachers are:

- why is this topic added (to the 1999 curriculum)
- what is the significance of knowing about these facts and theorems
- should these theorems be proved
- when do we use these theorems

Considering geometry learning merely as a process of knowing, proving and applying chains of theorems is not very helpful for many students. The above questions more or less reflect this common image of geometry learning. Reasoning in geometry should be more than logically connecting theorems. Making sense of geometrical objects and relations is the basis for good mathematical thinking in general, and rigorous deductive reasoning in particular. Like the development of other concepts in mathematics, there can be more emphasis in the early stage of recognizing and distinguishing notions that are originally vague but gradually more precise and well formulated in mathematical language. In Johnston-Wilder and Mason (2005), we can find many examples that explain this emphasis on certain important mathematical themes and pedagogical considerations throughout the process of developing geometric thinking. Regarding introduction of centers of triangle, an approach similar to that from the book is outlined below.

The starting key question is "what do we mean by a center of a triangle?" If we consider an equilateral triangle, which is regular and highly symmetric, there should be no problem in agreeing with where the center is. We can ask students to use simple drawing tools to add lines joining each vertex to its opposite side in order to locate the center more accurately. We can also describe the ways in which the center is related to different parts of the triangle (sides and vertices). Some possible descriptions are:

- the center is equidistant from the vertices
- the center is equidistant from the sides
- the center, when joining the vertices, will divide the triangle into 3 equal parts

These could be observed by students and articulated perhaps with the help of teacher. Similarly, the lines drawn, or imagined, by the students to locate the center bear certain characteristics and could be articulated in different ways. It is quite obvious that the lines form axes of symmetry. The notion of bisection can be highlighted and further explored. What exactly does each line bisect? Some possible suggestions are:

- the area is bisected
- the opposite side is bisected
- the angle is bisected

Note that a line that bisects a segment may only bisect its length or do so perpendicularly. On the other hand, a perpendicular from opposite vertex does not necessarily bisect a side. Of course, at this stage all these descriptions look the same regarding a center of an equilateral triangle. However it is useful to generate some of these descriptions before we realize that they apply to different centers or lines in other triangles. In fact, the next key question is "where could be the center for an ordinary scalene triangle". The students may notice that the notion of center is actually ambiguous in the general case. We may say that there are different centers depending on which description we stick to for defining a center, if one exists. That means, we may ask whether there is a point equidistant from the vertices or equidistant from the sides and understanding that these are different criteria.

At the same time, the meanings of these descriptions are developed, but not as a result of arbitrary definitions decided by mathematicians. After all, these special points are not merely brilliant discovery of geometers in the past. They could be well connected to our intuitive conceptions of center but gradually distinguished and formulated in more precise mathematical language. Being able to distinguish and articulate these properties, taking into consideration possible variation in the shape of triangles, is an important condition for deductive reasoning.

The use of extreme cases here can also be useful for illustrating the very different nature of these centers. For example, it is easy to notice that a circumcenter can be very far away from the vertices and lie outside a triangle, but without losing its essential property of being equidistant from the vertices. Similar to the previous discussion about centers of data, what makes a point qualified as a center does not depend on its appearance in individual cases, but related to what we are asking for. If we have to locate a point equidistant from the vertices, there is no reason to choose the incenter as a substitute just because the circumcenter lies outside a triangle. (Interestingly we tend to think that a mean is useless if it is far away from the median no matter what purpose we have in summarizing the data.) There can be application problems designed to suggest certain needs of locating points with some of these criteria, such as being equidistant from some points or lines. In these circumstances, the significance of such application problems is not only convincing students the practical use of these theorems. Instead, it could be a good opportunity for students to make distinction on their own and render explicit their assumptions about the geometric objects.

Starting from the Middle

There is a little booklet entitled "*Middles*" published by the Association of Teachers of Mathematics as a collection of puzzles related to the idea of middle in many different settings. The topics discussed in this article also come from an idea of middle from seemingly disconnected areas in our curriculum. There is also similarity in the teaching approach suggested, which may also be considered as teaching that "starts from the middle". It is the middle between two ends: one of which begins with mathematical content formally defined, while the other lies in the intuition and experiences of individual learners.

References

Baker, L. & Harris, I. (1995). *Middles.* Derby: Association of Teachers of Mathematics.

Curriculum Development Council (1999). *Syllabuses for Secondary Schools – Mathematics (Secondary 1–5).* Hong Kong: The Printing Department.

Johnston-Wilder, S. & Mason, J. (2005). *Developing Thinking in Geometry.* London: Paul Chapman Publishing.

Russell, S.J. & Mokros, J. (1996). *What Do Children Understand About Average?* Teaching Children Mathematics, Feb issue, pp.360–64.

Shaughnessy, J.M. (2006). Research on Students' Understanding of Some Big Concepts in Statistics. In G. Burrill, & P.C. Elliott (Eds). *Thinking and Reasoning with Data and Chance* (pp.77–98). Reston, VA: National Council of Teachers of Mathematics.

Watson, J.M. (2006). *Statistical Literacy at School: Growth and Goals. Studies in Mathematical Thinking and Learning.* Mahwah, N.J.: L. Erlbaum Associates.

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