Several approaches to a problem

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Find the maximum and minimum values of $z = x + y$ subject to $x^2 + y^2 = r$ where x and y are real.

It seems that this question can only be solved by Calculus. In fact, a high school student can tackle it. Usually, students would like to use Calculus to deal with problems involving finding maximum and minimum values. As a matter of fact, some sort of those problems can be solved by other methods such as the completing square method. The aim of this article is to illustrate other methods to deal with this type of question. Of course, this question is deliberately designed so that employing calculus is dispensable.

When $r = 0$, $x = y = 0$. Thus $z = 0$ only.

When $r < 0$, it is impossible to find real values of x and y. Hence, neither maximum value nor minimum value exists.

When $r > 0$, this question can be solved by the following methods:

Method 1 (Algebraic Method – using quadratic theory)

This problem can be solved by using the properties of roots and coefficients of a quadratic equation.

From $z = x + y$, we have $r = x^2 + y^2 = (x + y)^2 - 2xy = z^2 - 2xy$.

After transposing, we have $xy = \frac{1}{2}(z^2 - r)$. Now, we can form a quadratic equation in q whose roots are x and y:

$$q^2 - zq + \frac{1}{2}(z^2 - r) = 0$$

As this quadratic equation has real roots, its discriminant should be non-negative.

i.e. discriminant = $z^2 - 4 \times 1 \times (\frac{1}{2}(z^2 - r)) = z^2 - 2(z^2 - r) = 2r - z^2 \geq 0$

Hence, the range of z is $-\sqrt{2r} \leq z \leq \sqrt{2r}$.

Maximum value of z is $\sqrt{2r}$ and minimum value is $-\sqrt{2r}$.

"$x + y = z$ and $x^2 + y^2 = r$" leads me to think about the sum of roots and the product of roots. I have used this question in some of my capable classes of S4 and S6 as a challenging question. In my S4 science classes, this question was given to them after teaching quadratic equation and quadratic inequality. At the beginning, none of them could bring out any idea. Before the end of the lesson, I told them to find xy in terms of z and r . Next lesson, I noticed that some could form a quadratic equation whose roots are x and y but they did not proceed further. Maybe they did not realize the fact that the roots are real and thus, they could not think of using the discriminant. Another reason may be the expression involves a lot of "letters" which form a barrier and they could not go further then. At last, a few students could work out the problems. But in my S6 class, a few could work it out right away after telling them to express xy in terms of z and r .

Method 2 (Graphical Method)

The idea of this method comes from solving simple linear programming in two unknowns by graphical method (that is, by moving a straight line across the feasible region).

That is, the objective function is $z = x + y$ and the constraint is $x^2 + y^2 = r$ where $x, y \in \mathbf{R}$ and $r > 0$. The feasible "region" is the boundary of the circle $x^2 + y^2 = r$, where $r > 0$. We draw a line $x + y = c$ where c is a constant. Then we move the line $x + y = c$ across the circle in such a way that $x + y = c$ is parallel to $x + y = 0$. The first point at which the line $x + y = c$ enters the "region" is B while the last point at which the line $x + y = c$ leaves the "region" is A . (Note: We usually tell students the optimum solution can always be identified with one of the feasible corner points of the solution space. Now, we can treat the circle as a "polygon" with infinitely many sides.) Intuitively, it is not difficult to notice that the x-coordinate and the y-coordinate of A are the same. By simple calculation (refer to the appendix), the coordinates of A are $(\frac{\sqrt{2r}}{2}, \frac{\sqrt{2r}}{2})$. Similarly (refer to the appendix), the coordinates of B are $(\frac{-\sqrt{2r}}{2}, \frac{-\sqrt{2r}}{2})$. Hence the maximum value of $z = \frac{\sqrt{2r}}{2} + \frac{\sqrt{2r}}{2} = \sqrt{2r}$ and the minimum value of $z = \frac{-\sqrt{2r}}{2} + \frac{-\sqrt{2r}}{2} = -\sqrt{2r}$.

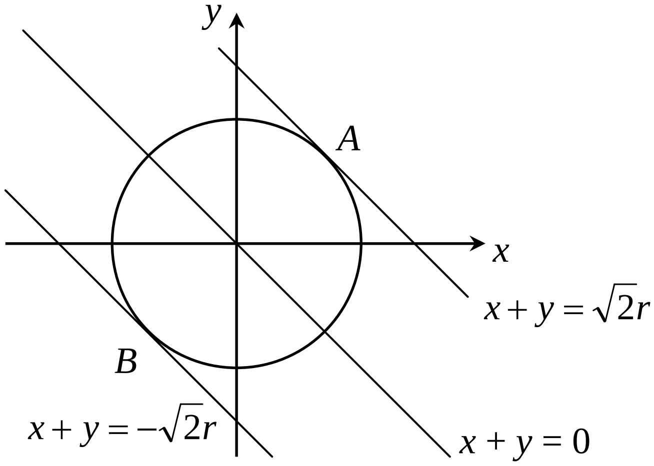

Figure 1

Method 3 (Trigonometric method)

As $x^2 + y^2 = r$ where $r > 0$ represents an equation of circle, we have the corresponding parametric form $x = \sqrt{r}\cos\theta$, $y = \sqrt{r}\sin\theta$. Then, $z = x + y = \sqrt{r}(\cos\theta + \sin\theta) = \sqrt{2r}\sin(\theta + \frac{\pi}{4})$. As $-1 \le \sin(\theta + \frac{\pi}{4}) \le 1$ for all real θ, $-\sqrt{2r} \le z \le \sqrt{2r}$. Hence, the results follow.

Reflection

1. I think that this question is suitable for any senior form students as a pastime problem. It is noticed that the knowledge needed to solve this problem does not go beyond the scope of S5 Mathematics or Additional Mathematics Curriculum.

2. I like to use it especially in lessons just before S5 students taking mock examination. Since it can involve a lot of topics, for instance, quadratic equation, quadratic inequality, linear programming, distance between a point and a straight line, equation of circle and subsidiary angles, it facilitates me to help students do revision during that period.

3. Teachers usually give their students homework that are related to the lessons just taught and students are expected to finish it in a few days. Teachers are also expected to mark them and give them back to students within a few days. Then, a new homework is assigned to students. The process goes on till the end of the academic year. Maybe such routine practice leads to cultivating a lot of students using a particular method for certain stereotyped problems. Thus, they seem to "resist" searching other methods. For example, "Prove that $6^n - 1$ is divisible by 5 for all positive integers n." It is found that most of students taking Additional Mathematics would like to use mathematical induction to prove this statement. In fact, this question can be solved by other methods. It is not difficult to find that the last unit of 6^n is always 6 and thus, the unit digit of $6^n - 1$ must be 5. Any integer with the unit digit being 5 must be divisible by 5. Moreover, this question does not confine us to use any particular methods. It is unnecessary to invoke mathematical induction to tackle this question.

Appendix

There are some methods to find the coordinates of A and B.

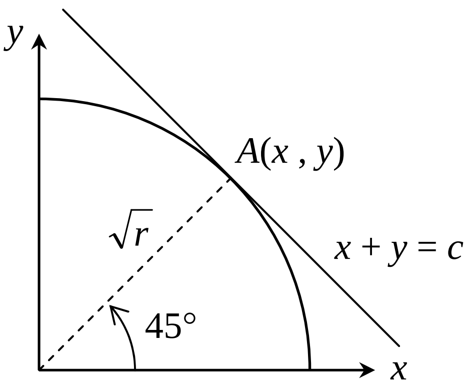

Figure 2

From figure 2,

$x = \sqrt{r}\cos 45° = \dfrac{\sqrt{2r}}{2}$ and

$y = \sqrt{r}\sin 45° = \dfrac{\sqrt{2r}}{2}$.

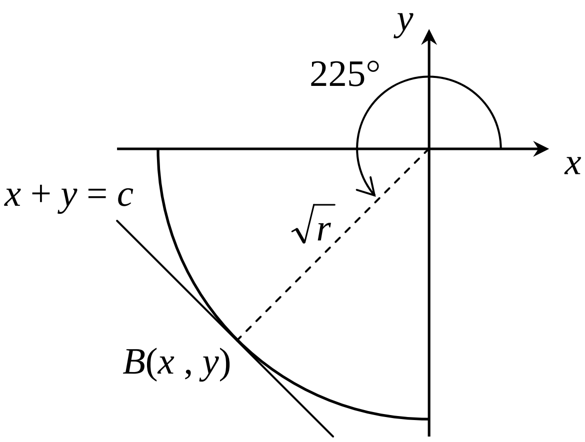

Figure 3

From figure 3,

$x = \sqrt{r}\cos 225° = \dfrac{-\sqrt{2r}}{2}$ and

$y = \sqrt{r}\sin 225° = \dfrac{-\sqrt{2r}}{2}$.

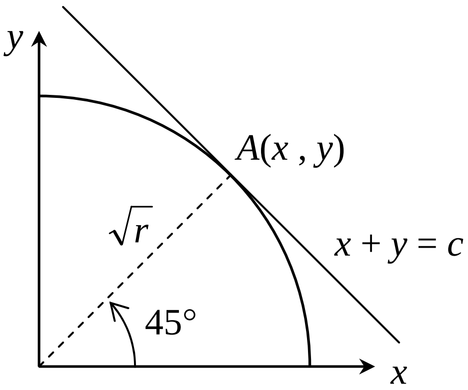

Figure 4

Alternatively, we have

$$\sqrt{r} = \frac{|\,0+0-c\,|}{\sqrt{2}}$$

Hence, $c = \pm\sqrt{2r}$.

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