Different Approaches to Solve a Plane Geometry Question

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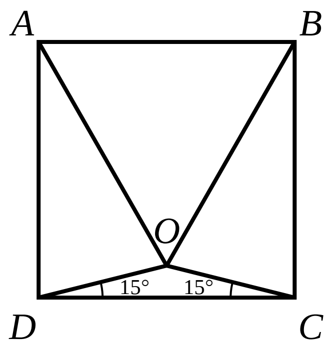
Introduction

Solving a plane geometry problem does not need to confine the approach using the theorems of plane geometry. It is natural to observe the layout of the plane figure to find symmetries, angles and triangles inside. It is a great pleasure to discover that various topics are connected in solving a simple problem. In addition, many difficult plane geometry problems could be solved by using Coordinate Geometry. The integration of knowledge from various topics is important for learning problem solving technique and asking meaningful questions. In this article, the author will illustrate different approaches to solving a plane geometry question that was discussed by C.M. Chung and W.M. Chiu in *EduMath* 9 (12/1999).

The Question

Given that $ABCD$ is a square and O is a point inside the square such that $\angle ODC = \angle OCD = 15^\circ$. Prove that ΔAOB is an equilateral triangle.

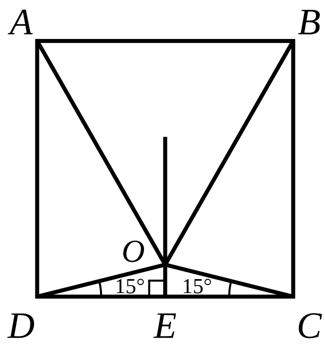


Approach 1 (Finding symmetries and observing angles and triangles inside)

It is a common technique to observe the symmetries of a plane geometry problem for getting useful results. It is no doubt that it applies to this problem as $\angle ODC = \angle OCD$ and the required result (ΔAOB is an equilateral triangle) suggest. The results of Lemmas 1 and 2, Theorems 1 and 2 could be derived easily by symmetry argument.

LEMMA 1 O must lie on the perpendicular bisector of DC.

PROOF As $\angle ODC = \angle OCD$, we could use symmetry argument. Alternatively, we could draw a perpendicular from O to DC, say at E. We could prove that $\Delta ODE \cong \Delta OCE$ by using $\angle ODC = \angle OCD$, $\angle OED = \angle OEC = 90°$ and the common side OE.



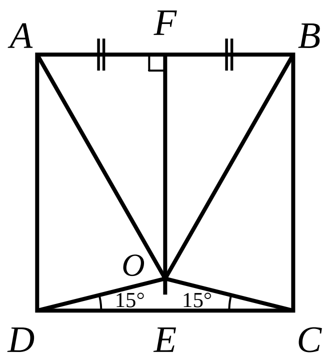
LEMMA 2 O must lie on the perpendicular bisector of AB.

PROOF As DC and AB are opposite sides of a square, the perpendicular bisectors of DC and AB are the same. Then result then follows by using Lemma 1.

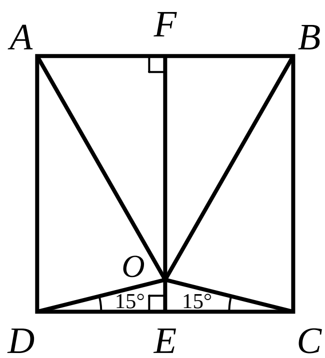
THEOREM 1 $OA = OB$.

PROOF By lemma 2, O lies on the perpendicular bisector of AB, so we could use symmetry argument.

Alternatively, we could draw a perpendicular bisector from O to AB, say at F. We could prove that $\Delta OFA \cong \Delta OFB$ by using $AF = BF$, $\angle OFA = \angle OFB = 90°$ and the common side OF. The result then follows, as OA and OB are the corresponding sides of the congruent triangles.



THEOREM 2 We could draw a line FOE such that $FOE \perp AB$ and $FOE \perp DC$ as shown in the diagram.



PROOF As DC and AB are opposite sides of a square, the perpendicular bisectors of DC and AB are the same. The result then follows by using Lemma 1.

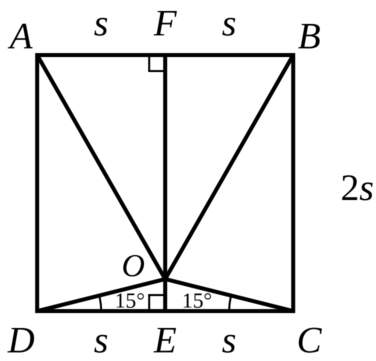
By Theorem 2, it is obvious that the problem is related to right-angled triangles ΔODE and ΔOAF. The use of trigonometric ratio and the Pythagoras' Theorem will be considered as usual.

LEMMA 3 $\tan 15° = 2 - \sqrt{3}$.

PROOF Using $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ and $\tan 30° = \dfrac{1}{\sqrt{3}}$, we know $\tan 15°$ is the positive real root of the equation $\dfrac{1}{\sqrt{3}} = \dfrac{2t}{1-t^2}$, i.e. $t^2 + 2\sqrt{3}\,t - 1 = 0$. The result follows from quadratic equation formula.

THEOREM 3 Let the length of the side of the square be $2s$. Then $OA = OB = AB = 2s$. That is, AOB is an equilateral triangle.

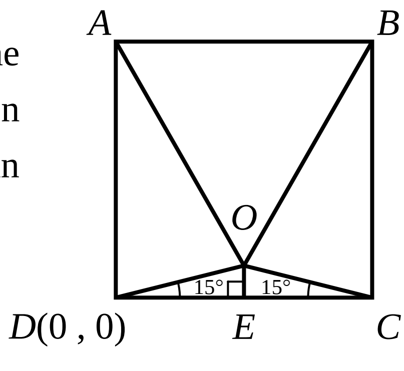
PROOF By Theorem 1, it suffices to show $OA = 2s$. By using Lemma 3 and considering ΔODE , we know $OE = s\tan 15° = (2-\sqrt{3})s$. Then we know $OF = EF - OE = \sqrt{3}\,s$. By considering ΔOAF and $OA^2 = AF^2 + OF^2$ (Pythagoras' Theorem) , we could get the result that $OA = 2s$.



Besides observing angles and triangles inside, we could consider the approach using Coordinate Geometry.

Approach 2 (Using Coordinate Geometry)

THEOREM 4 Let D be $(0, 0)$ and the length of the side of the square be $2s$. Then $OA = OB = AB = 2s$. That is, AOB is an equilateral triangle.

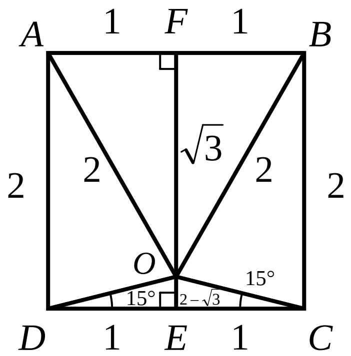


PROOF Obviously, A is $(0\,,2s)$, B is $(2s\,,2s)$ and C is $(2s\,,0)$. By using Lemma 1 and considering ΔODE , $DE = s$ and $OE = s\tan 15°$. Then O is $(s\,,s\tan 15°)$.

In order to show $OA = OB = AB = 2s$, by Theorem 1, it suffices to show $OA = 2s$. We could get this result from $OA = \sqrt{(s-0)^2 + (s\tan 15° - 2s)^2}$ and $\tan 15° = 2-\sqrt{3}$ (Lemma 3) .

Remarks

We have used the exact value of $\tan 15° = 2-\sqrt{3}$ to prove the desired result. In fact, the given diagram could be used to fine the exact value of tan 15 ° .



This may raise interests for students to further explore. For examples, how to find the exact value of sin 15° and cos 15° ? Could we get similar geometric constructions? Seeing the connection of different topics could aid the students to integrate the knowledge for problem solving and even asking meaningful questions for further discovery.