Evaluating the probability integral $\int_0^{\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x$ by elementary methods

Or Chi-Ming
Munsang College

Introduction

Here I would like to mention some elementary methods of evaluating the probability integral $\int_0^{\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x$. The result $\int_0^{\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x = \sqrt{\frac{\pi}{2}}$ is usually obtained by advanced techniques and is therefore omitted to sixth formers. In fact, with some plausible assumptions, it is possible to evaluate the probability integral using elementary methods which can be understood by students studying A-level Pure Mathematics. These methods deserve to be better known, as the result $\int_0^{\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x = \sqrt{\frac{\pi}{2}}$ is fundamental in the normal distribution of statistics, and is also a typical example of the surprising fact that although some ordinary integrals $\int_a^b f(x)\,\mathrm{d}x$ cannot be found, the improper integral $\int_0^{\infty} f(x)\,\mathrm{d}x$ can nevertheless be evaluated. The methods are modified as exercises here, since it would be more interesting for students to derive the result themselves than being told by teachers directly.

$\int_0^{\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x$ and $\lim_{n\to\infty}\sqrt{n}\int_0^{\frac{\pi}{2}}\cos^n\theta\,\mathrm{d}\theta$

The idea of Exercise 1 originated from [1] and [2]. Both authors of these two articles pointed out that the integral $\int_0^{\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x$ is closely related to

$\lim_{n\to\infty}\sqrt{n}\int_0^{\frac{\pi}{2}}\cos^n\theta\,\mathrm{d}\theta$. However, complicated integral transformations were used. In fact, these transformations can be avoided. Using the fact that $e^{-\frac{x^2}{2}}=\lim_{n\to\infty}\left(1-\frac{x^2}{2n}\right)^n$, the relation between $\int_0^{\infty}e^{-\frac{x^2}{2}}\,\mathrm{d}x$ and $\lim_{n\to\infty}\sqrt{n}\int_0^{\frac{\pi}{2}}\cos^n\theta\,\mathrm{d}\theta$ becomes more apparent, as shown in Exercise 1.

<u>Exercise 1</u>

Let $I_n=\int_0^{\frac{\pi}{2}}\cos^n x\,\mathrm{d}x$.

(a) For any positive integer $n\geq 2$, show that $I_n=\dfrac{n-1}{n}I_{n-2}$.

(b) Using (a), or otherwise, show that, for any positive integer $n\geq 2$,

$$n\,I_n\,I_{n-1}=(n-1)\,I_{n-1}\,I_{n-2}\ .$$

(c) Hence, or otherwise, show that, for any positive integer n,

$$n\,I_n\,I_{n-1}=\sqrt{\frac{\pi}{2}}\ .$$

(d) Using (c), or otherwise, show that $\lim_{n\to\infty}\sqrt{n}\,I_n=\sqrt{\dfrac{\pi}{2}}$.

(e) For any positive integer n, using the substitution $x=\sqrt{2nt}$, show that $\int_0^{\sqrt{2n}}\left(1-\dfrac{x^2}{2n}\right)^n\mathrm{d}x=\sqrt{2n}\int_0^1(1-t^2)^n\,\mathrm{d}t$.

Hence show that $\int_0^{\sqrt{2n}}\left(1-\dfrac{x^2}{2n}\right)^n\mathrm{d}x=\sqrt{2n}\,I_{2n+1}$.

By assuming that $\lim_{n\to\infty}\int_0^{\sqrt{2n}}\left(1-\dfrac{x^2}{2n}\right)^n\mathrm{d}x=\int_0^{\infty}\left(\lim_{n\to\infty}\left(1-\dfrac{x^2}{2n}\right)^n\right)\mathrm{d}x$, show that $\int_0^{\infty}e^{-\frac{x^2}{2}}\,\mathrm{d}x=\sqrt{\dfrac{\pi}{2}}$.

<u>*Brief Solution:*</u>

The result in (a) can be obtained by integration by parts. Using (a), we have $\quad n\,I_n \;=\; (n-1)\,I_{n-2}$

$$\Rightarrow \qquad n I_n I_{n-1} = (n-1) I_{n-1} I_{n-2}$$

$$\begin{aligned}\Rightarrow \qquad n I_n I_{n-1} &= (n-1) I_{n-1} I_{n-2} \\ &= (n-2) I_{n-2} I_{n-3} \\ &= \dots \\ &= 1 \cdot I_1 I_0 = \frac{\pi}{2}\end{aligned}$$

Since $I_n < I_{n-1}$, we have $n I_n^2 < n I_n I_{n-1} = \dfrac{\pi}{2}$.

Moreover, $n I_n^2 > n I_{n+1} I_n = \dfrac{n}{n+1}((n+1) I_{n+1} I_n) = \dfrac{n}{n+1} \cdot \dfrac{\pi}{2}$.

Therefore we have $\dfrac{n}{n+1} \cdot \dfrac{\pi}{2} < n\ I_n^2 < \dfrac{\pi}{2}$ and hence we have $\lim\limits_{n\to\infty} \sqrt{n}\, I_n = \sqrt{\dfrac{\pi}{2}}$ by squeezing theorem. Finally,

$$\begin{aligned}\int_0^{\sqrt{2n}} \left(1 - \frac{x^2}{2n}\right)^n \mathrm{d}x &= \sqrt{2n} \int_0^1 (1 - t^2)^n \, \mathrm{d}t \qquad , \text{by} \quad x = \sqrt{2n}t \\ &= \sqrt{2n} \int_0^{\frac{\pi}{2}} \cos^{2n+1} \theta \, \mathrm{d}\theta \qquad , \text{by} \quad t = \sin\theta \\ &= \sqrt{2n}\, I_{2n+1} \, .\end{aligned}$$

Taking limits, we have

$$\lim_{n\to\infty} \int_0^{\sqrt{2n}} \left(1 - \frac{x^2}{2n}\right)^n \mathrm{d}x = \lim_{n\to\infty} \sqrt{2n}\, I_{2n+1} = \sqrt{\frac{\pi}{2}}$$

$$\Rightarrow \int_0^{\infty} (\lim_{n\to\infty} \left(1 - \frac{x^2}{2n}\right)^n) \, \mathrm{d}x = \sqrt{\frac{\pi}{2}}$$

$$\Rightarrow \qquad \int_0^{\infty} e^{-\frac{x^2}{2}} \, \mathrm{d}x = \sqrt{\frac{\pi}{2}}$$

M. Spivak in [3] suggested another way of connecting the probability integral with $\sqrt{n} \int_0^{\frac{\pi}{2}} \cos^n \theta \, \mathrm{d}\theta$, which avoids using the assumption $\lim\limits_{n\to\infty} \int_0^{\sqrt{2n}} \left(1 - \dfrac{x^2}{2n}\right)^n \mathrm{d}x = \int_0^{\infty} (\lim\limits_{n\to\infty} \left(1 - \dfrac{x^2}{2n}\right)^n) \, \mathrm{d}x$. His approach is modified as Exercise 2 below.

Exercise 2

(a) Show that, for all real number x, $1-x^2 \le e^{-x^2} \le \dfrac{1}{1+x^2}$.

(b) Using (a), show that for any positive integer n,

$$\int_0^1 (1-x^2)^n \, \mathrm{d}x \le \int_0^1 e^{-nx^2} \, \mathrm{d}x < \int_0^\infty \frac{\mathrm{d}x}{(1+x^2)^n} \quad .$$

Hence, or otherwise, show that

$$\sqrt{n} I_{2n+1} \le \int_0^{\sqrt{n}} e^{-u^2} \, \mathrm{d}u < \sqrt{n} I_{2n-2} \quad \text{where} \quad I_n = \int_0^{\frac{\pi}{2}} \cos^n \theta \, \mathrm{d}\theta \quad .$$

(c) Using the result $\lim\limits_{n\to\infty} \sqrt{n} I_n = \sqrt{\dfrac{\pi}{2}}$ and (b), find $\displaystyle\int_0^\infty e^{-u^2} \, \mathrm{d}u$.

Brief Solution:

The result in (a) can be proved by differentiation easily. The first result in (b) is obtained by raising the inequality in (a) to power n and integrate, and note that $\displaystyle\int_0^1 \frac{\mathrm{d}x}{(1+x^2)^n} < \int_0^\infty \frac{\mathrm{d}x}{(1+x^2)^n}$. Then apply the substitutions $x = \sin\theta$, $u = \sqrt{n}x$ and $x = \tan\theta$ respectively to the three integrals to obtain the second result in (b). Take n tends to infinity, we have $\displaystyle\int_0^\infty e^{-u^2} \, \mathrm{d}u = \frac{\sqrt{\pi}}{2}$ by squeezing theorem.

Parametric Integration Technique

Another interesting method to evaluate the integral is the "parametric integration technique", which assumes that differentiation with respect to the parameter t of an integral can be carried under the integral sign ([4], [5]), as illustrated by Exercise 3.

Exercise 3

For any $t \ge o$, define $f(t) = \displaystyle\int_0^\infty \frac{e^{-t(x^2+1)}}{x^2+1} \mathrm{d}x$.

(a) Find $f(0)$.

(b) Show that $0 < f(t) \le e^{-t} \cdot \dfrac{\pi}{2}$.

Hence, or otherwise, find $f(\infty) = \lim_{t \to \infty} f(t)$.

(c) Assuming $\dfrac{d}{dt} f(t) = \int_0^\infty \dfrac{d}{dt}\left(\dfrac{e^{-t(x^2+1)}}{x^2+1}\right) dx$, show that

$$\frac{d}{dt} f(t) = -e^{-t} \int_0^\infty e^{-tx^2}\, dx \ .$$

Using the substitution $u = \sqrt{t}x$, show that

$$\frac{d}{dt} f(t) = -\frac{e^{-t}}{\sqrt{t}} I \quad \text{where} \quad I = \int_0^\infty e^{-u^2}\, du \ .$$

(d) By integrating the last result in (c) and using the result in (b), show that

$$I = \int_0^\infty e^{-u^2}\, du = \frac{\sqrt{\pi}}{2} \ .$$

<u>Brief Solution:</u>

$$f(0) = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} \ ,$$

$$f(t) = e^{-t} \int_0^\infty \frac{e^{-tx^2}}{x^2+1} dx$$

$$\le e^{-t} \int_0^\infty \frac{1}{x^2+1} dx = e^{-t} \cdot \frac{\pi}{2}$$

$$\Rightarrow \quad f(\infty) = 0 \ .$$

$$\begin{aligned}
\frac{d}{dt} f(t) &= \frac{d}{dt} \int_0^\infty \frac{e^{-t(x^2+1)}}{x^2+1} dx \\
&= \int_0^\infty \frac{d}{dt}\left(\frac{e^{-t(x^2+1)}}{x^2+1}\right) dx \\
&= \int_0^\infty - e^{-t(x^2+1)}\, dx \\
&= -e^{-t} \int_0^\infty e^{-tx^2}\, dx \\
&= \frac{-e^{-t}}{\sqrt{t}} \int_0^\infty e^{-(\sqrt{t}x)^2}\, d(\sqrt{t}x) \\
&= -\frac{e^{-t}}{\sqrt{t}} I \qquad \text{where} \quad I = \int_0^\infty e^{-u^2}\, du
\end{aligned}$$

Hence, $\quad f(\infty) - f(0) = -I\int_0^\infty \dfrac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t$

$$\Rightarrow \quad f(0) = I\int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t \qquad \text{as} \quad f(\infty) = 0$$

$$= 2I\int_0^\infty e^{-t}\,\mathrm{d}(\sqrt{t})$$

$$= 2I\int_0^\infty e^{-u^2}\,\mathrm{d}u \qquad \text{, by} \quad u = \sqrt{t}$$

$$= 2I^2$$

Hence $\quad I^2 = \dfrac{f(0)}{2} = \dfrac{\pi}{4}$

$$\Rightarrow \quad I = \frac{\sqrt{\pi}}{2}$$

The above exercises provide a good chance for students to learn an amazing and important result in analysis. Teachers can obtain more examples of evaluating other improper integrals in [4], [5] and [6], so that students can appreciate more beautiful results in analysis, and have more interesting exercises as well.

References

1. N. Gauthier, *Evaluating the probability integral*, The Mathematical Gazette V72.22 (1988), pp.124 – 125.
2. D. Desbrow, *Evaluating the probability integral*, The Mathematical Gazette V74.28 (1990), pp.169 – 170.
3. M. Spivak, *Calculus*, (Addison Wesley), (1967), p.329.
4. J. van Yzeren, *Moivre's and Fresnel's integrals by simple integration*, The American Mathematical Monthly V86 (1979), pp.691 – 693.
5. R.M. Young, *On the evaluation of certain improper integrals*, The Mathematical Gazette V75.4 (1991), pp. 88 – 89.
6. J. Wiener, *Differentiation with respect to a parameter*, The College Mathematics Journal V.32 (2001), pp. 180 – 181.